# Linear Algebra Lecture Videos

Dear MAT188 Students,

I’ve taught [MAT188](https://engineering.calendar.utoronto.ca/course/mat188h1) many times. The EdTech office generously video-captured many of my lectures to be shared with the MAT188 students. Although the course format has changed (from 13 weeks to 12, to a different book, etc.), I’m happy to share lecture videos from 2016 and 2017 with you in case they’re of any use.

We were using a different book in those years and so while I’ve tried to put the videos into a good order for you, some of the videos are Frankenvideos — bits of lectures from different lectures edited together. I’ve also referenced the lectures as closely as I could to the various sections of the textbook “[Linear Algebra with Applications](https://edtech.engineering.utoronto.ca/wp-content/uploads/sites/38/2024/02/Linear-Algebra-with-Applications-version-2021-revision-A-by-W.-Keith-Nicholson-2.pdf)” (version 2021, revision A) by [W. Keith Nicholson](https://profiles.ucalgary.ca/william-keith-nicholson). This book is an open textbook, provided by Lyryx.

The auto-subtitle feature on YouTube works surprisingly well, if you need subtitles. It’s not perfect but it’s pretty good if you need a little help parsing what I say in the video. *I misspeak on occasion and when this happened in class (and hence in the videos) I tried to point this out using italics in the comments below so that it’s clear that I misspoke. If you find what you believe to be unidentified mistakes, please email me and let me know.*

I love linear algebra and I hope you discover its beauty as well!

Sincerely,
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Professor, Mathematics

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## Lecture 1: An Introduction to Linear Systems

Nicholson, Section 1.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=umcQIX1fOiw) or [MyMedia](https://play.library.utoronto.ca/watch/c84b216f3107f4950a8c521fdc502987)

**Video Duration:** 26:50

### Video Description:

Introduction to some areas where linear algebra appears.

**0:45** Presented a “diet problem” which can be written as a linear system.

**7:00** Wrote the diet problem in terms of linear combinations of vectors.

**9:00** Solved the diet problem using methods from high school. (Note: you likely haven’t learnt about linear combinations of vectors yet but I hope that the explanation is clear enough that you can look past the language of “linear combinations” for the moment.)

**14:45** Wrote down a linear programming problem for the second diet problem (but didn’t solve it).

**17:13** Introduced the language of systems of linear equations (unknowns, coefficients, linear, right-hand sides). Presented a system of 2 equations in 2 unknowns.

**18:15** Discussed the system graphically and identified a solution of the system.

**21:35** Performed elementary operations on the system of linear equations and studied the new linear systems graphically.

## Lecture 2: Linear Systems: Solutions, Elementary Operations

Nicholson, Section 1.1

Watch the video on [YouTube](https://youtu.be/7lDzTs8Su_I) or [MyMedia](https://play.library.utoronto.ca/8c21c45be4375df1758325c3787e74e6)

**Video Duration:** 42:50

### Video Description:

Gave the definition of “a linear equation in n variables”.

**6:00** Defined what it means for a vector to be a solution of a linear equation.

**10:15** Note that if you write an equation like $x\_{1}-2x\_{2}+x\_{3}=1$ then [2;1;1] is a solution in R3. And [2;1;1;29] is a solution in R4. And [2;1;1;3;-8] is a solution in R5. An equation doesn’t determine the Rn that a solution lives in. Certainly, the equation won’t have solutions in R1 (what would you plug in for $x\_{2}$ and $x\_{3}$?) or solutions in R2 (what would you plug in for $x\_{3}$?) but it’s perfectly reasonable to consider that equation in R27 if needed - it depends on what physical problem the equation is coming from.

**10:25** Defined a “system of linear equations”.

**10:40** Stated that a linear system either has no solutions, has exactly one solution, or has infinitely many solutions. (The proof will come later!) Presented a linear system that has no solutions. Presented a linear system that has exactly one solution. Presented a linear system that has infinitely many solutions.

**16:00** Presented a system of 3 linear equations in 3 unknowns and found the solution using a sequence of elementary operations. (Swapping two equations, multiplying an equation by a nonzero number, adding a multiple of one equation to another equation.) Key in all of this is that the elementary operations don’t change the solution set. That is, if you have S = {all solutions of the original system} and T = {all solutions of the system you get after applying one elementary operation} then S = T. If you can prove that S = T for each of the three elementary operations then you know that the solutions of the original system are the same as the solutions of a later (easier to solve) linear system because you know that the solutions remain unchanged after each step on the way. (The worry is, of course, that you might lose solutions or gain solutions by doing these elementary operations. Certainly, if you multiplied one of the equations by zero you’d be at high risk of creating a new system that has more solutions than the previous system.)

**29:10** Presented a second 3x3 system and demonstrated that there were infinitely many solutions.

**31:20** Presented a third 3x3 system and demonstrated that there were no solutions.

**33:50** Gave an argument based on a specific system of 2 linear equations in 2 unknowns showing how it is that the elementary operations don’t change the set of solutions.

## Lecture 3: How to solve a system of linear equations

Nicholson, Section 1.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=dTL--uTAD8o) or [MyMedia](https://play.library.utoronto.ca/eafdbc2dae64e96bade050c3b44bf754)

**Video Duration:** 50:57

### Video Description:

**1:00** Defined the elementary (equation) operations.

**3:00** Defined what it means for two linear systems to be equivalent.

**6:30** Stated the theorem that equivalent linear systems have the same solution set.

**8:50** Introduced the augmented matrix as a short-hand way of representing a linear system of equations. Introduced the coefficient matrix $A$, the right-hand-side vector $\rightharpoonaccent{b}$. and the augmented matrix $(A|\rightharpoonaccent{b})$.

**9:30** Considered a specific linear system, represented it using an augmented matrix, introduced (and used) elementary row operations on the augmented matrix (which are the same thing as elementary (equation) operations) until the augmented matrix was in a simple form. Wrote down the linear system that corresponded to the simple augmented matrix and concluded that the original linear system had no solutions.

**19:20** Did another example in which it turned out there was exactly one solution.

**30:00** Introduced the language “leading ones” of a matrix that’s in reduced row echelon form.

**31:30** *I gave a wrong answer to a student - I claimed that if every student reduced the augmented matrix so that the first nonzero entry in a row is 1 then every student would have the same matrix. This isn’t true. What’s true is that if every student reduced the augmented matrix until it was in reduced row echelon form (the first nonzero entry in a row is 1 and above and below that 1 are zeros and the leading ones move to the right as you move down the rows and any zero rows are at the bottom of the matrix) then all students would have the same RREF matrix.*

**32:00** How to use MATLAB to find the RREF of a matrix.

**34:20** Did another example, this time there’re infinitely many solutions. How to write down the solution set. In the example, there are 3 equations and 4 unknowns. One of the unknowns is set to be a free parameter. Does it really matter which parameter you set to the free parameter? In this example, you could set $x\_{1}=t$ and find $x\_{2}=$ some expression of $t$, $x\_{3}=$ some expression of $t$. Alternatively, you could set $x\_{2}=t$ and find $x\_{1}=$ some expression of $t$, $x\_{3}= $some expression of $t$. Alternatively, you could set $x\_{3}=t$ and find $x\_{1}=$ some expression of $t$, $x\_{2}=$ some expression of $t$. It happens that the structure of the RREF of the augmented matrix is such that it’s easiest to do the $x\_{3}=t$ option.

**41:20** Introduced the concept of a “general solution” and then verified that it’s a solution, no matter what the value of the free parameter t.

**47:00** I went through the exercise of writing the solution when making the choice of setting $x\_{1}=t$ and find $x\_{2}=$ some expression of $t$, $x\_{3}=$ some expression of $t$. Hopefully, this was a gross enough exercise to convince you of the charm of choosing $x\_{3}=t$.

## Lecture 4: Reduced Row Echelon, Rank, Solutions

Nicholson, Section 1.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=QVLqCz6zhWg) or [MyMedia](https://play.library.utoronto.ca/7b5cad8d6b81e13d68d05b0e94283c45)

**Video Duration:** 46:04

### Video Description:

Started with a bird’s-eye view of how to solve a system of linear equations.

**5:10** First example is 3 equations with 4 unknowns - the augmented matrix is 3 x 5. If you had to make a bet, you’d guess that there’ll be infinitely many solutions. But don’t bet when you can find out the real answer. The rank of the augmented matrix is 3 while the rank of the coefficient matrix is 2. Because 2 < 3, there are no solutions to the linear system. (This is a fancy way of saying that when you write down the linear system corresponding to the RREF form of the augmented matrix, you get two equations that make sense but the third equation is 0=1. Which has no solution. Whenever rank(CoeffMatrix)<rank(AugmentedMatrix) this means that the linear system of the RREF form of the augmented matrix will have at least one equation of the form 0=1 and so there’s no solution.

**9:50** Another example with 3 equations with 4 unknowns. In this case, rank(CoeffMatrix) = rank(Augmented matrix) < number of unknowns. Because the two ranks are equal to one another, there’s at least one solution. Because the rank is less than the number of unknowns, there are infinitely many solutions and because “the number of knowns”- rank(AugmentedMatrix) = 2, the general solution has two free parameters.

**17:00** Wrote the general solution in vector form and discussed the three vectors in the general solution.

**23:55** Did another example, this one with 4 equations with 7 unknowns. The general solution has 4 free parameters.

**31:00** Defined the rank of a matrix and discussed it more fully.

**36:50** Stated the theorem about how the rank of the coefficient matrix and the rank of the augmented matrix determine whether there’s no, one, or infinitely many solutions.

*Missing lecture: The next thing in Nicholson’s book is Section 1.3 “homogeneous systems of linear equations; trivial and non-trivial solutions; linear combinations of solutions; basic solutions”. In the previous book and in the lectures from Fall 2016 and Fall 2017, these concepts were introduced but were interwoven with other material which you haven’t been introduced to yet. There’s no way to disentangle the material and so I have no lecture videos to offer you on the topic.*

## Lecture 5: Vectors, dot products, solutions of A*x*=*b*

Nicholson, Sections 4.1 & 4.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=zUv5akt0eMA) or [MyMedia](https://play.library.utoronto.ca/240ce8b50dfd99ec4d870b75f2b27b63)

**Video Duration:** 27:45

### Video Description:

Started with a quick review of Cartesian coordinates, vectors, vector addition, and scalar multiplication of vectors. The most important thing to keep track of is the difference between a point P(p1,p2) which has coordinates p1 and p2 and the position vector of this point P which is a vector whose tail is at the origin and whose tip is at P(p1,p2). In the course, we often use position vectors and points interchangeably and this can be very confusing sometimes.

**5:58 – 6:22** Ignore this part.

**6:22** Introduced the dot product for vectors in R2. First defined algebraically: the dot product of $\rightharpoonaccent{u}$= [u1;u2] and $\rightharpoonaccent{v}$ = [v1;v2] is u1\*v1+u2\*v2. Second, defined it geometrically: if you know the length of the two vectors and the angle between them then you can construct the dot product. (This leads to the natural question: if I gave a pair of vectors to two students and asked student A to compute the dot product using the algebraic definition and asked student B to compute the dot product using the geometric definition, would students A and B always get the exact same answer?) *The dot product can be defined in two ways and the two different ways of defining it give the same answer and are useful in different ways. This is a powerful and important aspect of the dot product, not discussed in the book.*

**10:00** Generalized the dot product and length to vectors in Rn.

**12:10** Introduced what it means for two vectors to be orthogonal. Note! The zero vector is orthogonal to all vectors.

**19:40** Stated the theorem that tells us how the dot product interacts with scalar multiplication and with vector addition.

**23:30** Multiplying a matrix and a vector to verify a solution of a system of linear equations.

## Lecture 6: Introduction to Planes

Nicholson, Section 4.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=KevlLHJjVIM) or [MyMedia](https://play.library.utoronto.ca/c5fc6adccba6e1435e6b95faeba4a200)

**Video Duration:** 15:06

### Video Description:

Reminded students of Cartesian Coordinates. Language: given a point P(p1,p2) in the Cartesian plane, it will be represented using a position vector $\rightharpoonaccent{P}$ whose tail is at the origin O(0,0) and whose head is at the point P(p1,p2).

**4:00** Introduced the vector representation of a plane through the origin in R3. *I used the language of subspaces - the span of two vectors - you don’t know this yet. Just ignore that and keep going…*

**6:28** Introduced the vector representation of a plane through the point P(p1,p2,p3) in R3. How the vector representation of a plane is related to the scalar representation of a plane.

**8:30** Given a plane that’s not through the origin, consider a point P(p1,p2,p3) that’s in the plane. Does this mean that the point’s position vectors *p* lies in the plane? No! But if P(p1,p2,p3) and Q(q1,q2,q3) lie in the plane then the vector $\rightharpoonaccent{q}$-$\rightharpoonaccent{p}$ will be parallel to the plane.

**10:10** How to find the scalar equation for a plane from a normal vector to the plane and a point in the plane.

## Lecture 7: Projection onto a vector, Projection perpendicular to a vector

Nicholson, Section 4.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=wtnRn65UCcM) or [MyMedia](https://play.library.utoronto.ca/0c24c0048e92897c2dff13e80ddf3e96)

**Video Duration:** 47:51

### Video Description:

Review of definition of the dot product of two vectors.

**4:02** Projection of the vector $\rightharpoonaccent{y}$ onto the vector $\rightharpoonaccent{x}$. Presented a derivation of $Proj\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$ using SOHCAHTOA and the geometric definition of the dot product.

**12:30** Example using two specific vectors.

**15:34** Introduced $Perp\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$ and showed how to find it once you’ve found $Proj\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$.

**22:45** Verify that $Proj\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$ is parallel to $\rightharpoonaccent{x}$ and that $Perp\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$ is orthogonal to $\rightharpoonaccent{x}$. Verify that $Proj\_{\rightharpoonaccent{x}}\left(\rightharpoonaccent{y}\right)+Perp\_{\rightharpoonaccent{x}}\left(\rightharpoonaccent{y}\right)=\rightharpoonaccent{y}$.

**25:30** Presented a second derivation of $Proj\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$; this derivation is based on the algebraic definition of the dot product.

**32:20** Find the distance from a point to a plane. This is where we really need to be careful about the difference between a point P(p1,p2,p3) and its position vector $\rightharpoonaccent{p}$.

## Lecture 8: Review of projections, Introduction to the cross product

Nicholson, Section 4.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=rmX2lQkz-lg) or [MyMedia](https://play.library.utoronto.ca/1ebbead2886550adfd101a498a2525c3)

**Video Duration:** 49:52

### Video Description:

Started with a correction to a mistake in previous lecture then did a review of $Proj\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$ and $Perp\_{\rightharpoonaccent{x}}(\rightharpoonaccent{y})$.

**7:00** Showed that $Proj\_{\rightharpoonaccent{x}}\left(Perp\_{\rightharpoonaccent{x}}\left(\rightharpoonaccent{y}\right)\right)=\rightharpoonaccent{0}$.

**25:20** Started with cross products. The dot product of two vectors works for any two vectors in Rn. The cross product of two vectors only works for vectors in R3. The cross product of two vectors in R3 is a vector in R3.

**27:45** It’s possible to generalize cross products in some sense - for example, given 4 vectors in R5 there’s a way of using them to create a 5th vector in R5. This is analogous to a cross product on R5.

**30:00** Given a vector equation of a plane, find a scalar equation of the plane. This means that you need to find a normal vector to the plane. This can be done using the cross product of two vectors that are parallel to the plane. (Or it can be done by solving a system of 2 linear equations in 3 unknowns…) Verified that the cross product is orthogonal to the vectors that created it.

**42:00** Presented the formula for how to compute the cross product of two vectors in R3.

**47:00** The cross product of a vector with itself is the zero vector. Showed that $\rightharpoonaccent{u}$ x $\rightharpoonaccent{v}$ = - $\rightharpoonaccent{v}$ x $\rightharpoonaccent{u}$. Proved that $\rightharpoonaccent{u}$ $∙$ ($\rightharpoonaccent{u}$ x $\rightharpoonaccent{v}$) = 0.

## Lecture 9: Properties and Uses of the Cross Product

Nicholson, Section 4.3

Watch the video on [YouTube](https://www.youtube.com/watch?v=qHEbN9CF05U) or [MyMedia](https://play.library.utoronto.ca/333da18af71f3cc56c2d7517d82b9105)

**Video Duration:** 48:23

### Video Description:

**1:55** The properties of the cross product: scalar multiplication, vector addition, anti-symmetry.

**5:00** Find a scalar equation for the plane that contains three given points. (Apologies for the video – the camera person wasn’t following the blackboards as well as usual…)

**16:30** Introduced ||$\rightharpoonaccent{u}$ x $\rightharpoonaccent{v}$ || = ||$\rightharpoonaccent{u}$|| ||$ \rightharpoonaccent{v}$|| sin(theta) where theta is the angle between $\rightharpoonaccent{u}$ and $\rightharpoonaccent{v}$. Included a discussion of why it’s sin(theta) and not |sin(theta)|.

**21:12** How the cross product is related to the area of a parallelogram.

**25:08** How to use a dot product and a cross product to compute the volume of a parallelepiped.

## Lecture 10: Matrix Addition, Scalar Multiplication, Transposition

Nicholson Section 2.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=yeeBY7niYLU) or [MyMedia](https://play.library.utoronto.ca/45e9ccbc865bf841eddba3bec0f5f6f1)

**Video Duration:** 21:02

### Video Description:

Up to now, matrices have been used as a form of short-hand for solving systems of linear equations. Now we’re going to start doing algebra with matrices - adding matrices, multiplying matrices, and so forth. To do this, I started by introducing the language of matrices in terms of entries. Defined square matrices, upper triangular matrices, lower triangular matrices, diagonal matrices.

**15:00** How to add matrices, how to multiply a matrix by a scalar (i.e. a real number in this class).

**20:00** Defined the transpose of a matrix and discussed its properties.

## Lecture 11: Matrix Transformations

Nicholson Section 2.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=7OI7aIY9-eA) or [MyMedia](https://play.library.utoronto.ca/6bc5c0eb1b346d11a848f9f0953c6f10)

**Video Duration:** 28:32

### Video Description:

*Warning: the lectures came from a book that referred to “matrix mappings”, “linear mappings”, and “linear operators”. Your book refers to “matrix transformations”, “linear transformations” and “geometrical transformations”. So whenever I say the word “mapping” you should think “transformation”.* Started matrix mappings w/ a review of language from high school: function, domain, range, etc. Introduced language of “codomain”.

**6:55** Given an mxn matrix A, define the matrix transformation (aka matrix mapping) TA (aka fA).

**12:50**  Did two examples of matrix transformations where the matrices A are both 2x2 matrices.

**15:30** Showed how to represent the matrix transformation graphically via “before” and “after pictures”. Note that from the graphic representation it’s clear that TA($\rightharpoonaccent{x}$+$\rightharpoonaccent{y}$) = TA($\rightharpoonaccent{x}$) + TA($\rightharpoonaccent{y}$).

**21:00** For the first example, does it appear that the matrix mapping is onto R2?

**21:35** Did second example, discussing its graphic representation. It’s clearly not going to be onto R2.

## Lecture 12: Introduction to Matrix Multiplication

Nicholson Section 2.3

Watch the video on [YouTube](https://www.youtube.com/watch?v=UwZ7xcw5ufo) or [MyMedia](https://play.library.utoronto.ca/6809db6039658d50d12fb647fb2f994e)

**Video Duration:** 13:46

### Video Description:

Started with a review of an earlier example that motivated matrix-vector multiplication.

**3:50** Introduced matrix multiplication. Just because AB is defined, doesn’t mean that BA is defined.

**4:50** And even if they’re both defined, it doesn’t mean that AB=BA. AB and BA might not even be the same size.

**9:15** In general, matrix multiplication doesn’t commute - the order matters! Even if AB and BA are the same size.

**11:00** Did an example showing what happens when you multiply a matrix by a diagonal matrix. This is an important example to remember.

**12:10** For real numbers you know that ab = ac implies b=c only if a is nonzero. Similarly, if AB=AC this doesn’t always imply that B=C. Gave an example of a nonzero matrix A such that AB=AC but B doesn’t equal C.

## Lecture 13: Introduction to Matrix Inverses

Nicholson Section 2.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=KNQpAGnLWaY) or [MyMedia](https://play.library.utoronto.ca/ad155a47abe054a3a1cf3384e7209eed)

**Video Duration:** 47:53

### Video Description:

Started by reminding students that $A\rightharpoonaccent{x}=\rightharpoonaccent{b}$ will have a) no solution, b) exactly one solution, or c) infinitely many solutions and discussed what this had to do with the Column Space of A and rank(A). *You don’t know what the Column Space of A is yet - so ignore that part!*

**4:40** One option to trying to solve $A\rightharpoonaccent{x}=\rightharpoonaccent{b}$ is via elementary row operations. Discussed the costs & benefits of this approach.

**6:20** Another option is to find a matrix B (if it can be found) so that AB = I (the identity matrix) and use B to find the solution x. Discussed the costs & benefits of this approach.

**12:50**  When is it better to use elementary row operations to try and solve $A\rightharpoonaccent{x}=\rightharpoonaccent{b}$ and when is it better to try and find B so that BA=I?

**16:00** Does every square matrix have some matrix so that BA = I? Gave an example of a 2x2 matrix for which there is no B so that BA=I - presented two different arguments as to why there could never be a B so that BA=I.

**25:00** Presented a super-important and super-useful theorem about matrix inverses.

**31:20** Used theorem to construct an algorithm to try and find a matrix B so that BA=I. Note: the algorithm is using block multiplication. Specifically, if A is 2x2 and B is 2x2 with B = [$\rightharpoonaccent{b1}$;$\rightharpoonaccent{b2}$] then AB = A [$\rightharpoonaccent{b1}$;$\rightharpoonaccent{b2}$] = [$A \rightharpoonaccent{b1}$;A $\rightharpoonaccent{b2}$] = [$\rightharpoonaccent{e1}$ ; $\rightharpoonaccent{e2}$] = I. Make sure that you’re comfortable with this block multiplication!

**37:45** Used the matrix inversion algorithm on a 2x2 matrix for which there is a B so that BA=I.

**40:50** Used the matrix inversion algorithm on a 2x2 matrix for which there isn’t a B so that BA=I.

**43:30** Bird’s eye view of matrix inversion algorithm.

**46:20**  Defined what it means for a square matrix to be invertible.

## Lecture 14: Properties of Inverse Matrices, Invertible Matrix Theorem

Nicholson Section 2.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=NUPoOONC6Bk) or [MyMedia](https://play.library.utoronto.ca/c1b1333b95b9b5b671d8f1af039585e7)

**Video Duration:** 47:08

### Video Description:

Halloween lecture: instructor was mugged and replaced by angel.

Started with the definition of “invertible matrix”. Reviewed the super-useful theorem which the matrix inversion algorithm is based on.

**2:00** Simple example: when is a diagonal matrix invertible?

**5:50** Three 3x3 matrices - are the invertible? Note that if the first matrix is A then the second matrix is 2A and the third matrix is AT. From the example, we suspect that if t is nonzero then (t A)-1 = 1/t A-1 and if A is invertible then (AT)-1 = (A-1)T.

**11:45** Stated theorem to this effect. The proof of the theorem uses the super-important theorem: if you can find C so that AC = I then voila - you’re done - you’ve shown that A is invertible, you’ve found the inverse of A, and you get to write A-1 = C. (You don’t even get to write A-1 until you’ve shown A is invertible.) Basically, if you can find a matrix that does the job that an inverse should do then you’ve found the inverse.

**28:00** Did a classic exam question: if A is a square matrix such that A2 – A = 2 I, find A-1.

**31:45** Inversion does not “play well” with matrix addition.

**38:40** The Invertible Matrix Theorem. Given a square matrix, presented 6 equivalent statements. If any one of them is true then all of them are true and it follows that the matrix is invertible. The point: some of those statements are pretty easy to check!

Note: item 5 is that the columns being linearly independent is something that you haven’t learnt about yet - ignore this item for the moment. Item 6 is that the Column Space of A is Rn - you haven’t learnt about the Column Space of a matrix - ignore this item for the moment.

*Here's* [*a nice video on Linear Transformations and how to present linear transformations using matrices (i.e. as a matrix transformation)*](https://www.youtube.com/watch?v=kYB8IZa5AuE&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=4&ab_channel=3Blue1Brown) *by 3Blue1Brown. It’s really worth watching even if we can’t present the graphics this way in class, in the book, or on the exams…*

*The previous book focused more on linear transformations than the current book. And so there’re more lectures on the subject than you likely saw. Either the extra lectures below are helpful for you or they’re overkill…*

## Lecture 15: Introduction to Linear Transformations

Nicholson Section 2.6

Watch the video on [YouTube](https://www.youtube.com/watch?v=x9pzxkLqwpM) or [MyMedia](https://play.library.utoronto.ca/53b2cc38e0ae851fb8c0300f4b0e2cd6)

**Video Duration:**

### Video Description:

If TA is a matrix transformation from R2 to R2 and I tell you what TA does to the vector [1;0] and what it does to the vector [0;1] can you use this to figure out what TA does to any vector [x1;x2] in R2?

**6:00** How matrix transformations act on the sum of two vectors, how matrix transformations act on the scalar multiple of a vector.

**7:45** I define “linear transformation” (aka “linear mapping”). Example of a transformation from R2 to R2 which is not a linear transformation.

## Lecture 16: Geometric examples of Linear Transformations

Nicholson Section 2.6/Section 4.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=sGNhtQ0GGBg) or [MyMedia](https://play.library.utoronto.ca/790a27fca3d61a0ffa02b8e2895a7f22)

**Video Duration:** 48:48

### Video Description:

Started by reminding definition of linear transformations. Review of the “before” and “after” presentation of what a transformation from R2 to R2 does. (There’s nothing special about this before and after way of presenting what a transformation does - it’s just that it’s easiest to draw when the domain and codomain are in R2 because I can draw everything easily rather than trying to draw things in R3 or R5 or something…)

**4:25** In terms of this “before” and “after” presentation, what does it mean for T($\rightharpoonaccent{x}$+$\rightharpoonaccent{y}$) = T($\rightharpoonaccent{x}$) + T($\rightharpoonaccent{y}$)?

**7:49** In terms of this “before” and “after” presentation, what does it mean for T(r $\rightharpoonaccent{x}$) = r T($\rightharpoonaccent{x}$)?

**8:43** Is the mapping T([x1;x2]) = [1;x2] a linear transformation? We can see pictorially that it isn’t. Separately, we can check that T([2;0]+[0;2]) doesn’t equal T([2;0]) + T([0;2]). (To show that something isn’t a linear transformation you just have to give a single example where it breaks one of the rules.)

**11:56** Started geometric transformations. First example: dilation. Proved it’s a linear transformation.

**19:55** Can I represent dilation as a matrix mapping? Yes, but need to be careful - what matrix you get depends on what basis you use for the domain. (One thing that’s confusing/important is that when I defined dilation, it was done w/o referring to any specific basis - it was defined simply as “given a vector in R2, double its length”. I didn’t need to refer to the coordinates of the vector - the moment I refer to the coordinates of the vector I’ve implicitly chosen a basis. For example, if I’m in matlab and I write “$\rightharpoonaccent{x}$=[2;3]” then implicit in this is the standard basis and what I mean is “$\rightharpoonaccent{x}$ = 2 [1;0] + 3 [0;1] = 2 $\rightharpoonaccent{e1}$ + 3 $\rightharpoonaccent{e2}$”.)

**32:08** Second example: “silly putty” transformation. Note: represented this transformation requires some sort of coordinates because it does one thing in one direction and nothing in another. And so I define it directly in terms of coordinates (I’ve implicitly chosen a basis like {[1;0],[0;1]}). Checked that the transformation is linear and represented it as a matrix transformation.

**40:10** Third example: shear transformation. Represented it using coordinates (you can check on your own that it’s a linear transformation) and represented it as a matrix transformation.

## Lecture 17: Representing Linear Transformations as Matrix Transformations

Nicholson Section 2.6/Section 4.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=sPOI3HLbbO8) or [MyMedia](https://play.library.utoronto.ca/f575dc217cd3031d4b857cb8af1aeb88)

**Video Duration:** 50:39

### Video Description:

Started with a discussion of language mapping/transformation/operator.

**2:00** Revisited shear mappings. Shear in x direction: shear to the right versus shear to the left. Shear in y direction: shear up versus shear down.

**13:37** Does shearing change area?

**16:35** The linear transformation T($\rightharpoonaccent{x}$) = Proj\_[2;1]($\rightharpoonaccent{x}$). Discussed domain, codomain, range, vectors that’re sent to the zero vector by the linear transformation, vectors that’re unchanged by the transformation. All of this was done intuitively; want to do it rigorously.

**37:42** The transformation that corresponds to reflecting a vector in a given line. Referred students to the book on how to understand this transformation in terms of projections, how to represent it as a matrix transformation and so forth. (Basically, you need to do all the stuff that was done in 16:35-37:41 but for this new transformation.)

**39:16** The transformation that corresponds to rotating a vector counter-clockwise by a given angle.

**47:07** Composition of geometric transformation - how to find a matrix transformation that represents the composition of three geometric transformations. (Note: the same logic would apply for the composition of any number of geometric transformations, not just three. And it’s not limited to geometric transformations; it works for linear transformations in general.)

## Lecture 18: Composition of Linear Transformations

Nicholson Section 2.6/Section 4.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=CyY5g5jVurk) or [MyMedia](https://play.library.utoronto.ca/4a4583f3651f63b6939402e84eaf2497)

**Video Duration:** 25:01

### Video Description:

Started with the composition of two geometrical transformations. How to find the matrix transformation that represents the composition of two linear transformations.

**13:00** Is rotating and then shearing the same as shearing and then rotating? What’s a fast way to answer this question?

## Lecture 19: Introduction to Determinants

Nicholson Section 3.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=F2yEzwhbzIo) or [MyMedia](https://play.library.utoronto.ca/9ef20068e0e43cccea36e34b51e6652a)

**Video Duration:** 40:57

### Video Description:

Is the general 2x2 matrix A = [a,b;c,d] invertible? Used the matrix inversion algorithm on this general 2x2 matrix and found that in order for to be invertible we need ad-bc to be nonzero. Also, if ad-bc is nonzero then there’s a formula that we can memorize that gives us the inverse of A.

**7:55** Stated a theorem for 2x2 matrices about whether or not they’re invertible.

**8:30** Defined the determinant of a 2x2 matrix.

**12:00** Defined the determinant of a 3x3 matrix using a formula*. I do not have this formula memorized even though I’ve been using and teaching linear algebra for over 30 years. The reason I could write it down so quickly is because I was looking at the matrix and writing it down by knowing the definition in terms of cofactors (see 15:55 for 3x3 matrices and 29:10 for general NxN matrices) and applying that definition in real time.*

**14:25** For a general 3x3 matrix: if the third row is a multiple of the second row, showed that the determinant is zero. (You should make sure that you can also repeat this argument if the second row is a multiple of the third row.)

**15:55** Noted that the determinant of a 3x3 matrix is found using a specific linear combination of determinants of 2x2 submatrices.

**17:45** General discussion of computing determinants of 4x4 matrices, 5x5 matrices, 6x6 matrices - how many 2x2 determinants will be needed? Computing a determinant’s a lot of work! (Will there be a faster way? We’ll see in the next lecture that there is.)

**21:10** Defined the determinant of an NxN matrix in terms of a cofactor expansion along the first row. Defined what a cofactor is.

**25:45** Computed the determinant of a specific 3x3 matrix.

**29:00** How to use wolfram alpha to find the cofactors of a square matrix; you can use this to check your work.

**31:55** Stated a theorem that states that the determinant of a square matrix can be computed by using a cofactor expansion about any row or column - it doesn’t have to just be the first row. You can choose whatever row or column that makes it easiest for you.

**33:50** Demonstrated the usefulness of this theorem by computing the determinant of an upper triangular matrix.

**38:05** Theorem: If A is upper triangular or lower triangular or diagonal matrix then det(A) is the product of the diagonal entries of A.

**39:25** Gave a 3x3 motivation for det(A) = det(AT).

## Lecture 20: Elementary row operations and determinants

Nicholson Section 3.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=_TwtL6UGCRs) or [MyMedia](https://play.library.utoronto.ca/b0d2b2cf46be635d8bea3a84645e1090)

**Video Duration:** 50:56

### Video Description:

**1:40** Computed the determinant of a specific 3x3 matrix by doing a cofactor expansion about its second column.

**6:00** Did elementary row operations on A and carried it to Row Echelon Form. Computed the determinant of the REF matrix. *When computing determinants you don’t need to carry the matrix to RREF, just to REF! Once it’s in REF, you can use the fact that the determinant of an upper triangular matrix is the product of the diagonal entries.*

**9:30** Stated the effect of each elementary row operation on the determinant of a matrix and explained how to remember these rules.

**14:05** Revisited the previous example and figured out how to figure out the det(A) from the determinant of the REF matrix as long as you know the sequence of elementary row operations you took to get to the REF. If *all* you have is A and the REF matrix then you can’t find det(A) from the determinant of the REF matrix.

**18:10** Is there any reason to carry A all the way to RREF? Did this for an example and showed that it still works but it’s an unnecessary amount of work if all you want is det(A). The real point of this example was to show why if det(A) is nonzero then the RREF of A must be the identity matrix (and therefore A is invertible). And if det(A) is zero then the RREF of A must have a row of zeros (and therefore A is not invertible).

**31:25** Proved that if you create B by multiplying a row of A by t then det(B) = t det(A). Did this for a 3x3 matrix.

**38:00** Proved that if create B by swapping two rows of A then det(B) = - det(A). I proved this by induction. I proved that it’s true for 2x2 matrices by using the definition of determinant. I showed how to leverage this knowledge about 2x2 matrices to prove that it’s true for 3x3 matrices. The next step is to leverage this knowledge about 3x3 matrices to prove that it’s true for 4x4 matrices. This goes on forever and this idea is the idea behind proof by induction.

**48:40** Used the theorem to show that if A has a repeated row then det(A) = 0.

## Lecture 21: Usefulness of the determinant: invertibility and geometry

Nicholson Section 3.2/Section 4.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=cAc8v3YXFbw) or [MyMedia](https://play.library.utoronto.ca/505d7b07cdd514e5f4e5766d9ef57d0e)

**Video Duration:** 49:15

### Video Description:

Review of how the three elementary row operations affect the determinant.

**6:50** If A is equivalent to B by some sequence of elementary row operations then det(B) equals some *nonzero* number times det(A). It follows that A is invertible if and only if det(A) is nonzero.

**16:30** How determinants interact with products of square matrices: det(AB) = det(A)det(B). It follows that det(AB) = det(BA) and that if A is not invertible then AB and BA aren’t invertible either.

**19:00**  How to remember that det(AB) = det(A)det(B).

**21:00**  Did a classic midterm question involving determinants.

**27:00** How to use determinant to compute the area of a parallelogram. Discussed why the absolute value is needed.

**35:00**  How the area of the image of a region under a linear mapping is determined by the determinant of the standard matrix for the linear mapping and the area of the region. *The previous book introduced “standard matrix” early on which is why I’m referring to in these lectures; Nicholson only introduces it in chapter 9. So you don’t know this language. Here’s what “standard matrix”* [*means*](https://math.stackexchange.com/questions/313798/find-the-standard-matrix-for-a-linear-transformation)*. Nicholson refers to “the matrix of a linear transformation” at the bottom of page 106. This is the “standard matrix”; he just doesn’t call it that until page 497 (he’s trying to avoid confusing you too early, I assume).* I proved this for a parallelogram and stated it for general regions in the plane. Note: the proof for general regions in the plane is a multivariable Calculus thing, not a linear algebra thing.

**41:00**  Example where the linear mapping is rotation.

**43:30**  Example where the linear mapping is reflection about a line.

**47:00**  Example where the linear mapping is projection onto a line.

## Lecture 22: Powers of matrices, introduction to eigenvalues & eigenvectors

Nicholson Section 3.3

Watch the video on [YouTube](https://www.youtube.com/watch?v=xG5g1-XdntU) or [MyMedia](https://play.library.utoronto.ca/7a9b4ad9f047444d0fb880613a2167ea)

**Video Duration:** 47:44

### Video Description:

Started with a 2x2 matrix and looked at what happened if I applied it over and over again to [1;0]. It’s converging to the vector [3;-2]. What’s with that? If I apply it over and over again to a different vector, I find that the result converges to something. Why? How did I compute A40 power anyway?

**5:25** Represented the matrix as a product of three matrices, one of which is diagonal. This made it super-easy to compute An and also to figure out where those limiting vectors were coming from.

**16:20**  Introduced the definition of an eigenvector of a linear mapping. Defined eigenvalue and eigenvector-eigenvalue pair.

**18:40**  Did geometric example - what are the eigenvalues & eigenvectors for reflecting about a line? What do they mean geometrically?

**23:10**  Demonstrated that a nonzero multiple of an eigenvector is also an eigenvector.

**27:20**  Why do we require that eigenvectors be nonzero? 29:15 Did geometric example - what are the eigenvalues & eigenvectors for projecting onto a line? What do they mean geometrically?

**33:00**  What about counterclockwise rotation by theta? Can you find a (real) eigenvector?

**34:40**  If I give you a matrix and a vector, how can you figure out if the vector is an eigenvector? If it is an eigenvector, how can you find its eigenvalue?

**37:35**  Given a matrix, how do I find its eigenvectors and eigenvalues? Tried the natural first idea - tried to find the eigenvector vector and eigenvalue simultaneously. Got two nonlinear equations in three unknowns. Yikes!

**41:15** Try to break the problem into two steps. First find the eigenvalues. Subsequently, for each eigenvalue try to find eigenvectors. Explained why we’re looking for lambdas so that det(A-lambda I) = 0.

## Lecture 23: How to find eigenvalues and eigenvectors

Nicholson Section 3.3

Watch the video on [YouTube](https://www.youtube.com/watch?v=dVv08T9N_F8) or [MyMedia](https://play.library.utoronto.ca/40624f6c1217b06a5bbe4141a6b56689)

**Video Duration:** 49:24

### Video Description:

*Note: In this lecture, I use the language of “linear combinations” - you haven’t seen this language yet but I hope it’s clear enough what is meant.* Reviewed definition of eigenvector and eigenvalue.

Reviewed why we’re looking for lambdas so that det(A-lambda I) = 0.

**6:30** Returned to the reflection example from previous lecture. We know the eigenvalues & eigenvectors geometrically but how could we have found them algebraically? Worked through the example.

**16:45**  Important! What happens if you’d made a mistake when you computed your eigenvalues? What happens when you then try to find eigenvectors?

**21:30** Did a 3x3 example. Here we don’t have geometric intuition and we’re going to have to compute the eigenvalues by finding the roots of a cubic polynomial. This example’s interesting because we get a repeated eigenvalue and so when we look for eigenvectors we get them in two different directions.

**44:10**  Important! If you add two eigenvectors together and they have different eigenvalues, is the sum also an eigenvector? No!

**46:15** Slammed through a final 3x3 example, introduced the language of “algebraic multiplicity” of eigenvalues. In this example, there was a repeated eigenvalue but I couldn’t find two eigenvectors w/ different directions.

## Lecture 24: Introduction to Diagonalization

Nicholson Section 3.3/Section 5.5

Watch the video on [YouTube](https://www.youtube.com/watch?v=DShojAkAi2I) or [MyMedia](https://play.library.utoronto.ca/58ea94c83dc51f79c4e991d223f294f6)

**Video Duration:** 50:11

### Video Description:

*Note: in this lecture I use the language of “linear independence”. You haven’t seen this yet. For two vectors, it means that they’re not parallel. Hold onto that concept and you’ll learn more about linear independence later on.*

Started by reviewing definition of eigenvalue and eigenectors.

**3:50** Reviewed procedure for finding eigenvalues and eigenvectors of A.

**8:00** Given a 3x3 matrix w/ eigenvalues 1,2,2 what are the benefits? (Only need to solve two linear systems when hunting eigenvectors.) What are the risks? (You might not be able to find two fundamentally different eigenvectors when you’re looking for eigenvectors w/ eigenvalue 2.)

**11:00** Considered a 3x3 matrix A and built a matrix P out of three eigenvectors of A. Computed AP, using block multiplication. The result is that AP is a matrix where each column is a multiple of the corresponding column of P. What is the multiple? The eigenvalue. The result is the P diag(labmda1, lambda2,lambda3) where lambda1, lambda2, and lambda3 are eigenvalues of A. *Make sure you understand this portion of the lecture very well - all of diagonalization is built on AP = P diag( …).*

**20:00** Introduced the concept of diagonalization. It's critical that the matrix P be invertible.

**23:10** What happens if I change the order of the columns in P? The new matrix will still be invertible. What happens to AP? What’s the result on the diagonal matrix that I ultimately get.

**27:00** What happens if I replace one of the columns of P with a nonzero multiple of that column?

**32:20** What happens if I replace one of the columns of P with a nonzero multiple of one of the other columns?

**36:30** Defined what it means for a square matrix to be diagonalizable. *Note: my definition (that A = P diag(…) inv(P) ) is different from the one in your book (that inv(P) A P is diagonal matrix) but the two definitions are equivalent if you left- and right-multiply by the appropriate matrices.*

**37:40** Stated a theorem that if A is diagonalizable then the columns of P are eigenvectors and the entries in the diagonal of the diagonal matrix are eigenvalues.

**42:45** Given a specific 3x3 matrix, is it diagonalizable? Did the long division on the characteristic polynomial, in case you’re wanting to see that. If you don’t want to spend so much ink/lead then learn how to do synthetic division; this is short-hand for long division.

## Lecture 25: Introduction to Systems of Linear ODEs

Nicholson Section 3.5

Watch the video on [YouTube](https://www.youtube.com/watch?v=vq_mA944c9c) or [MyMedia](https://play.library.utoronto.ca/ac4e609a8cfc10c15c4295c44a44d9fb)

**Video Duration:** 49:37

### Video Description:

*The lecture starts with a crash course on ordinary differential equations. If you don’t understand a single ODE then you’ve got no chance of understanding a system of ODEs… That said, if you’re short on time and don’t have time for the “why bother?” aspect then just go directly to* ***38:00****.*

**0:50** Started with a simple ODE problem; modelling a tank of liquid with an inflow and an outflow.

**7:15** What is a differential equation? What is a solution? What does it mean for a function to satisfy a differential equation?

**11:20** How to find a differential equation that models the saltwater tank?

**17:40** Presented the solution of the initial value problem. Answered the questions about the solution.

**20:30** Now need to address the question about what’s the right flow rate to use. This involves going back to the original modelling equation and introducing a parameter for the flow rate and finding a solution that depends on both the parameter and on time. (Before the flow rate was just a number and the solution only depended on time.)

**24:45** Introduced the “smell the coffee and wake up” problem. Rooms in an apartment are linked to one another by air vents. Presented the problem and the questions one has about the problem.

**33:45** How to write the pair of ODEs that model the case with only two rooms.

**38:00** Wrote the system of two ODEs as an ODE for a vector in R2; a 2x2 matrix A is involved. Gave a bird’s eye recipe for the approach.

**39:50** The eigenvectors and eigenvalues of A.

**40:40** Wrote down the general solution (didn’t explain where it came from).

**41:30** What we need to do to satisfy the initial condition on the ODE. *Note: something that sometimes confuses students is what happens when there’s a zero eigenvalue. In this case, exp(0 t) = 1, and so your solution involves a vector that doesn’t change in time; it just sits there. In this case, the solution has a vector 50 [1;1] in addition to a vector that does depend on time 50 exp(-2/500 t) [1;-1].*

**43:45** Analyzed the solutions to make sure that they make sense - Do you get what you expect as time goes to infinity?

**45:00**  Presented the solutions using matlab. This allowed me to play with the more general problem, including varying the number of rooms, the flow rate, etc. You see that if there are enough rooms then people in the far rooms won’t smell enough coffee to be woken by it. *If you’d like the matlab script, send me an email and I’ll send it to you!*

## Lecture 26: Systems of Linear ODEs — where do the solutions come from?

Nicholson Section 3.5

Watch the video on [YouTube](https://www.youtube.com/watch?v=C2vEKb_-VMU) or [MyMedia](https://play.library.utoronto.ca/ead6e84a144d2a15f51d63d46c705a37)

**Video Duration:** 44:11

### Video Description:

Started by reminding students of what the system of ODEs is.

**2:00** Last class I presented the general solution as a linear combination of things. First, I demonstrate that each of these things solve the system of ODEs. Translation: I demonstrate that if (lambda,$\rightharpoonaccent{v}$) is an eigenvalue-eigenvector pair of A then
 $\rightharpoonaccent{x}\left(t\right)=$exp(lambda t) $\rightharpoonaccent{v}$ is a solution of the system of ODEs.

*Note: I blithely differentiate vectors but you’re probably not so happy doing that - you’re used to differentiating single functions. You need to sit down and write the time-dependent vectors in terms of their components and convince yourself that differentiating the vector is the same as differentiating each component and that the things I do so quickly (like d/dt of exp(lambda t)* $\rightharpoonaccent{v}$ *equals lambda exp(lambda t)* $\rightharpoonaccent{v}$*.*

**9:30** I demonstrate that if $\rightharpoonaccent{x1}\left(t\right)$ and $\rightharpoonaccent{x2}\left(t\right)$ are two solutions of the system of ODEs and c1 and c2 are constants then c1 $\rightharpoonaccent{x1}\left(t\right)$ + c2 $\rightharpoonaccent{x2}\left(t\right)$ is also a solution of the system of ODEs.

**17:30**  Given k eigenvalue-eigenvector pairs, I can write a solution of the system of ODEs that involves k coefficients c1, c2, … ck.

**18:20**  Choose the coefficients using the initial data.

**21:00**  If we have a problem in Rn, what happens if k doesn’t equal n? Do we need k to equal n? Answer: if I’m going to be able to solve every possible initial set of initial conditions I’m going to need n linearly independent eigenvectors in Rn. (*You don’t know what linear independence means yet - sorry! Come back to this in a few weeks. In the meantime, it would have sufficed to say “If A is a diagonalizable matrix then…”*)

**25:45**  Presented a different argument for where the general solution came from. This argument relies on diagonalizing A. We reduce a problem that we don’t know how to solve to a problem that we do know how to solve.

*Looking for lectures on linear operators? See lectures 11-14!*

## Lecture 27: Introduction to linear combinations

Nicholson Section 5.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=jkHI1jDyqkc) or [MyMedia](https://play.library.utoronto.ca/c4b56bd002a076b3504963ff5d75c269)

**Video Duration:** 22:18

### Video Description:

Introduced set notation (relatively) slowly and carefully. Introduced R2, Z2.

**5:15** Did an example where a set is graphically presented as a collection of vectors in R2.

**11:08** Scalar multiplication of vectors, introduced linear combinations.

**17:50**  Introduced the word “basis” but didn’t carefully define it.

**18:30** Introduced “the standard basis for R2”, R3, and R4.

## Lecture 28: Subsets, Subspaces, Linear Combinations, Span

Nicholson Section 5.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=5re30wZ8JCY) or [MyMedia](https://play.library.utoronto.ca/8ee941672755a5059c483ed7f5b344df)

**Video Duration:** 46:27

### Video Description:

Introduce language “subspace” and “span”.

**1:45** Introduced 6 subsets of R2 using set notation.

**4:48** Presented each subset as a collection of vectors in R2. Note: I was not as careful as I could have been - when presenting one of the quadrants (Q2, for example) I simply shaded in a region of the plane as if it were a collection of points. Really, I should have drawn one position vector for each point in that region, filling up the quadrant with infinitely many vectors of varying lengths and angles (but with the angles always in a certain range, determined by the subset Q2).

**21:30** Defined what it means for a subset of Rn to be a subspace of Rn.

**22:00** Went through the previously introduced 6 subsets and identified which were subspaces.

**24:50** Introduced a subset S of R4; is it a subspace? Proved that it’s a subspace of R4.

**33:44** Given a set of vectors in R3 - if one lets S be the set of all linear combinations of these vectors, what would this set look like in R3?

**35:20** Another subset of R3. Is it a subspace?

**37:27** A subset of R2. Is it a subspace?

**42:45** Another subset of R2. Is it a subspace?

**43:35** Given {$\rightharpoonaccent{v1}$,$\rightharpoonaccent{v2}$,..$\rightharpoonaccent{vk}$} a set of k vectors in Rn, introduced Span($\rightharpoonaccent{v1}$,$\rightharpoonaccent{v2}$,..$\rightharpoonaccent{vk}$) and stated that it’s a subspace of Rn.

## Lecture 29: Linear Dependence, Linear Independence

Nicholson Section 5.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=6Qn7KjzgMlk) or [MyMedia](https://play.library.utoronto.ca/91dbaeefccb48749d104b2d4bfa0d8ac)

**Video Duration:** 58:21

### Video Description:

**1:30** Introduced a set which is the set of all linear combinations of two specific vectors in R2. Discussed the set, examples of vectors in the set, etc.

**10:00** Theorem: the set of all linear combinations of two (fixed) vectors in Rn is a subspace of Rn. Proved the theorem.

**19:00** Introduced Span($\rightharpoonaccent{v1}$,$\rightharpoonaccent{v2}$,…$\rightharpoonaccent{vk}$) and stated that it’s a subspace of Rn.

**22:00** Gave an example of a subset of R2 which is closed under scalar multiplication but not under vector addition. Gave an example of a subset of R2 which is closed under vector addition but not under scalar multiplication.

**24:40** Example in which I show that [7;1;13] is not in Span([2;1;4],[-1;3;1],[3;5;7]). Used high school methods to solve the problem. NOTE: there’s a mistake at **30:00**. The second equation should be 6 t1 + 10 t3 = 20, not 2 t1 + 4 t3 = 6!! When plugging t1 into the third equation, one gets -2 t3 + 83/7 = 13, which has the solution t3 = -4/7. This then determines t1 = 30/7 and t2 = -1/7 and, in fact, we have that [7;1;13] is in Span([2;1;4],[-1;3;1],[3;5;7]) because (30/7)\* [2;1;4]+(-1/7)\*[-1;3;1]+(-4/7)\*[3;5;7] = [7;1;13]. What went wrong? I made a mistake when copying from my notes. My notes had the example “[7;1;13] is not in Span([2;1;4],[-1;3;-1],[3;5;7]).” Note that the third component of the second vector in the span is -1, not +1, as written on the blackboard. If you correct that mistake then you’ll get that the rest of the approach works and that you end up with the impossibility of 44/7=6.

**33:30** Example in which I show that [7;0;13] is in Span([2;1;4],[-1;3;-1],[3;5;7]). I simply wrote down the solution without showing how to find it. Students were then asked to demonstrate that [7;0;13] is in Span([2;1;4],[-1;3;-1]) and [7;0;13] is in Span([2;1;4],[3;5;7]) and [7;0;13] is in Span([-1;3;-1],[3;5;7]).

**40:00** Proved that Span([2;1;4],[-1;3;-1],[3;5;7]) equals Span([2;1;4],[-1;3;-1]).

**43:30** Defined what it means for a set of vectors to be linearly independent.

**45:00** Defined what it means for a set of vectors to be linearly dependent. Gave an example of a linearly dependent set of four vectors in R3.

**48:06** Showed that {[2;1;2],[-2;2;1],[1;2;-2]} is linearly independent.

## Lecture 30: More on spanning and linear independence

Nicholson Section 5.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=ThMWxTnNVDQ) or [MyMedia](https://play.library.utoronto.ca/b3b4a6f6035a2be5db216c5774a7e0b2)

**Video Duration:** 50:52

### Video Description:

Started by reminding students of definition of span and linear independence.

**4:48** Is [3;-1;2;1] in Span([1;1;0;1],[2;0;0;2],[0;2;-1;1])?

**14:10** Does Span([1;1;2],[1;-1;-1],[2;1;1]) equal R3? Can an arbitrary vector [a;b;c] be written as a linear combination of the three vectors?

**25:45** Is {[1;1;2],[1;-1;-1],[2;1;1]} a basis for R3?

**29:40** Is a set of 4 vectors in R3 a linearly independent set? Asked about a specific example but it should be clear to you that whenever you have 4 or more vectors in R3 the set will be linear dependent. This is because when you set up the system of linear equations needed to address the question you’ll have more unknowns than you have equations. This means that either there’s no solution or there’re infinitely many solutions. (Having exactly one solution is not an option.) And having no solution isn’t an option (because you already know that the zero vector is a solution) so it follows that there are infinitely many solutions.

**34:09** Stated this as a general theorem.

**36:40** Considered a specific example of 3 vectors in R3. Are they linearly dependent or not? It turned out they aren’t.

**41:00** Considered the same set of 3 vectors - do they span R3? No. What’s interesting is - in the process of figuring out that the answer is “no” you find the scalar equation of the plane that the three vectors do span.

**46:45** Given 3 vectors in R3, do they form a basis for R3? (answer maybe yes, maybe no. It depends on the specific vectors.) What about 2 vectors? (answer: never! Can’t span R3!) What about 4 vectors (answer: never! Will always be linearly dependent!)

## Lecture 31: Bases, finding bases for Rn

Nicholson, Section 5.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=orseal90kQI) or [MyMedia](https://play.library.utoronto.ca/0037d08ebc9cad90bc79a9307ec18cc8)

**Video Duration:** 1:01:52

### Video Description:

Started with a review of the definitions of linear independence and basis.

**1:50** Need to consider the one bizarre subspace of Rn. The subset S = {$\rightharpoonaccent{0}$} is a subspace of Rn. But it has no basis because no set of vectors in S will be a linearly independent set. This discussion is the reason why theorems in the book (like Theorem 5.2.6) assume that the subspace in question is not {$\rightharpoonaccent{0}$}.

**5:30** Is a given set of 4 vectors in R3 a basis for R3? No: demonstrated that the vectors are linearly dependent.

**12:00** Discussed theorem that says that if you have k vectors in Rn and you build an n x k matrix A by putting the vectors into the columns of A then: the k vectors are linearly independent if and only if rank(A) = k. Discussed the implications of the theorem.

**17:20** Given any m x n matrix A, if you find its RREF then rank(A) = number of leading 1s in RREF à rank(A) $\leq $ number of columns of A. But we also know rank(A) = number of nonzero rows of the RREF à rank(A) $\leq $ number of rows of A. So rank(A) $\leq $ min{m,n}. This is a super-important fact that we use all the time.

**23:15** Two vectors in R3. Can they be a basis for R3? No: demonstrated that they cannot span R3.

**34:00** Discussed theorem that says: given k vectors in Rn, if k < n then the vectors cannot span Rn.

**37:00** Is a given set of 3 vectors in R3 a basis for R3? Yes: it turns out to be a basis. Given n vectors in Rn, then figuring out whether or not it’s a basis takes work - there’s no fast answer.

**42:34** I wrote "k vectors in Rn form a basis for Rn if and only if the rank of the coefficient matrix equals k." This is ridiculously wrong. If I have two vectors {<1,0,0>,<0,1,0>} in R3 then clearly they aren't a basis for R3. But for this example, k=2 and the rank of the coefficient matrix is 2. So by the (wrong) theorem I wrote up, those two vectors form a basis for R3.

**43:00** Stated and discussed theorem that says that k vectors will span Rn if rank(A)=n where A is the n x k matrix whose columns are the vectors in question.

**46:45** Up to this point in the lecture, all I was working on was whether or not a set of vectors was a basis for Rn. At this point, I turned to a proper subspace, S, of R4 and sought a basis for the subspace S. (“Proper subspace of R4” means a subspace which is smaller than R4.)

**52:25** Corrected mistake from last class.

**55:25** In general, given k vectors in R4, can they be a basis for R4? Discussed this for k<4, k=4, and k>4. Make sure that you understand this argument and that it has nothing to do with R4 - if the vectors were in Rn you’d be looking at three cases k<n, k=n, and k>n.

## Lecture 32: Subspaces Related to Linear Transformations

Nicholson Section 5.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=y97dGttLy4U) or [MyMedia](https://play.library.utoronto.ca/170019f114981a6977b10889753a626f)

**Video Duration:** 23:20

### Video Description:

Introduced subspaces that are related to a linear transformation L from Rn to Rm and subspaces that are relevant to an mxn matrix A. Null(L) is a subspace of Rn, Range(L) is a subspace of Rm, Null(A) = “Solution Space of Ax=0” is a subspace of Rn, Col(A) = “span of all columns of A” is a subspace of Rm, Row(A) = “span of all rows of A” is a subspace of Rn.

**5:00** If [L] is the standard matrix for a linear transformation L, is there any connection between the two subspaces Null(L) and Range(L) and the three subspaces Null([L]), Col([L]), and Row([L])? Answer: Null(L)=Null([L]) and Range(L)=Col([L]). *The previous book introduced “standard matrix” early on which is why I’m referring to in these lectures; Nicholson only introduces it in chapter 9. So you don’t know this language. Here’s what “standard matrix”* [*means*](https://math.stackexchange.com/questions/313798/find-the-standard-matrix-for-a-linear-transformation)*. Nicholson refers to “the matrix of a linear transformation” at the bottom of page 106. This is the “standard matrix”; he just doesn’t call it that until page 497 (he’s trying to avoid confusing you too early, I assume).*

**7:15** Proved Range(L) is a subspace of Rm.

**19:45** Did example for L from R3 to R3 where L projects a vector $\rightharpoonaccent{x}$ onto the vector [1;2;3]. Found Range(L) and Null(L) using geometric arguments.

## Lecture 33: Introduction to Solution space, Null space, Solution set

Nicholson Section 5.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=e6HiZDS7d6g) or [MyMedia](https://play.library.utoronto.ca/2ab7b5b31d973df57de201224297ba3d)

**Video Duration:** 48:37

### Video Description:

**1:26** Lecture starts. Given a matrix A, defined the “solution space of Ax=0” and, given a linear transformation L from Rn to Rm, defined Null(L).

**6:20** Proved that Null(L) is a subspace of Rn.

**16:00** Example of L:R3🡪R3 where L corresponds to projection onto a specific vector. Found Null(L) by geometric/intuitive arguments. Verified the intuition by representing the linear transformation as a matrix transformation and finding the corresponding solution space. Found a basis for Null(L). Found the dimension of Null(L).

**31:45** Example of L:R3🡪R3 where L corresponds to projection onto a specific plane. Found Null(L) by geometric/intuitive arguments. Verified the intuition by representing the linear transformation as a matrix transformation and finding the corresponding solution space. Found a basis for Null(L). Found the dimension of Null(L).

**41:45** Given an mxn matrix A, defined the solution set of the system $A\rightharpoonaccent{x}=\rightharpoonaccent{b}$. Found solution set for a specific 2x4 matrix A. *This is an important example because A has a column of zeros and students often get confused by such matrices!* Wrote the solution set as a specific solution of $A\rightharpoonaccent{x}=\rightharpoonaccent{b}$ plus a linear combination of solutions to the homogeneous problem$A\rightharpoonaccent{x}=\rightharpoonaccent{0}$.

## Lecture 34: Introduction to range of a linear transformation, column space of a matrix

Nicholson Section 5.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=y-BBQmpIKTE) or [MyMedia](https://play.library.utoronto.ca/97a8b82cf1c556fa0fc079d719fa7278)

**Video Duration:** 50:55

### Video Description:

L is a linear transformation from Rn to Rm and A is a mxn matrix. Started with a review of Null(L) and the solution space of $A\rightharpoonaccent{x}=\rightharpoonaccent{0}$; these are subspaces of Rn.

**2:00** Introduced the range of a linear transformation and the column space of a matrix. These are subspaces of Rm.

**5:30** Found Range(L) where L is the linear transformation corresponding to projection onto a specific vector. Once we found Range(L), then we found a basis for Range(L).

**21:30** Found Range(L) where L is the linear transformation corresponding to projection onto a specific plane. Once we found Range(L), then we found a basis for Range(L).

**37:55** Defined the column space of a matrix: Col(A).

**39:10** Given a specific matrix, A, found a basis for Col(A).

## Lecture 35: Null(L), Range(L), the rank theorem, Row(A)

Nicholson Section 5.4

Watch the video on [YouTube](https://www.youtube.com/watch?v=kBMmd_t9Hjg) or [MyMedia](https://play.library.utoronto.ca/c507dc15a8103bc9f3e41140adb82670)

**Video Duration:** 50:55

### Video Description:

Started with a review of the previous lecture: Null(L) and Range(L) for a pair of geometric transformations from R3 to R3. For both examples, dim(Null(L)) + dim(Range(L)) = 3. This will be true in general - if L goes from Rn to Rm we’ll have dim(Null(L)) + dim(Range(L)) = n.

**7:55** Given L from Rn to Rm, I presented an algorithm for finding a basis for Null(L).

**15:45** Given L from Rn to Rm, I presented an algorithm for finding a basis for Range(L).

**22:40** Stated the rank theorem dim(Null(L))+dim(Range(L)) = n and explained why it’s true.

**25:00** Defined the “nullity” of a linear transformation/the “nullity” of a matrix.

**28:40** Defined the row space of A: Row(A). It’s a subspace of Rn.

**31:40** Stated the theorem that if A and B are row equivalent then Row(A)=Row(B). This means that when you do elementary row operations on a matrix, the row space of the final matrix is the same as the row space of the initial matrix is the same as the row space of each intermediary matrix.

**32:45** Applied the theorem to a classic exam question.

**43:30** Discussed a classic mistake that confused/under-rested students make when asked for a basis for Col(A).

**45:10** Did an example of finding a basis for the solution space S = {x such that Ax=0}.

## Lecture 36: Introduction to Sets of Orthogonal Vectors

Nicholson Section 5.3/Section 8.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=GNWnE7Psxxc) or [MyMedia](https://play.library.utoronto.ca/ec3a119c912bb9c438acad938c2dd4dd)

**Video Duration:** 53:20

### Video Description:

Defined what it means for a set of vectors to be orthogonal. Demonstrated that if P is a matrix whose columns are mutually orthogonal then PT P = diagonal matrix.

**4:30** Started with a discussion of change of basis material that wasn’t discussed and what students aren’t responsible for.

**5:45** Recalled the definition of an orthogonal set of vectors. Discussed specific examples.

**9:28** Why do we care? A) If a set is orthogonal it’s easy to check if it’s linearly independent. B) If a set’s orthogonal it’s easy to figure out whether a specific vector is in the span of the set. C) If a matrix is symmetric then it’s diagonalizable and you can find an orthogonal basis of eigenvectors.

**12:30** Proved that an orthogonal set of nonzero vectors is linearly independent.

**14:45** *Misspoke and said “orthogonal” instead of “linearly independent”.*

**21:20** If a vector $\rightharpoonaccent{v}$ is in the span of a set of orthogonal vectors, here’s a fast way of finding the linear combination that equals $\rightharpoonaccent{v}$.

**26:30** Stated a theorem which is “Given an orthogonal set of vectors, if $\rightharpoonaccent{v}$ is in the span of the orthogonal set then you can immediately write down a linear combination that equals $\rightharpoonaccent{v}$.

**28:30** What goes wrong if $\rightharpoonaccent{v}$ isn’t in the span of the orthogonal set of vectors.

**31:45** What goes wrong if the set of vectors isn’t orthogonal. Gave an algebraic explanation and a geometric explanation of what goes wrong.

**39:45** Given an orthogonal set, what’s a fast way of checking if a vector $\rightharpoonaccent{v}$ is in their span?

**45:45** Review of wonderful properties of orthogonal sets of vectors. What’s the cost of this “free lunch”?

**48:15** Defined what it means for a vector to be orthogonal to a subspace S of Rn. Gave a geometric example of S and vectors that are orthogonal to S.

## Lecture 37: More on diagonalization

Nicholson Section 5.5

Watch the video on [YouTube](https://www.youtube.com/watch?v=O5-bMU2dvk0) or [MyMedia](https://play.library.utoronto.ca/1e3efa1756142288bf02ff8ae3c6aaff)

**Video Duration:** 48:40

### Video Description:

Started by reviewing the definition of what it means for a square matrix to be diagonalizable.

**1:20** An nxn matrix A is diagonalizable if and only if you can find n linearly independent eigenvectors of A. *Note: this lecture uses the language of linear independence, which you haven’t learnt yet. Your book refers to “basic eigenvectors” and rather than asking that you have a full set of linearly independent eigenvectors, it asks that you have as many “basic eigenvectors” as the algebraic multiplicity of the eigenvalue. It’s the same thing but if you find this all too confusing then just skip this lecture or return to it after you’ve learnt about linear independence.*

**5:30** Defined what it means for two square matrices to be “similar”. Reviewed a 3x3 example.

**13:19** Stated theorem “Any set of eigenvectors with distinct eigenvalues is a linearly independent set”.

**14:55** Compared the eigenvalues of A to the eigenvalues of the diagonal matrix. (They’re the same.) Compared the trace of A to the trace of the diagonal matrix. (They’re the same.) Compared the determinant of A to the determinant of the diagonal matrix. (They’re the same.)

**18:40** Returned to another prior 3x3 example.

**25:45** Did a new 3x3 example. This one is super-important it’s a “nearly diagonal” matrix and it’s not diagonalizable.

**34:14** Stated theorem that if A and B are similar matrices then they have the same eigenvalues, same determinant, same trace, and same rank. *Note: this material isn’t in this section of Nicholson’s book. You’re not responsible for it, I just left it in because it’s important and beautiful and the proofs require you to understand some important things. But feel free to skip directly to* ***44:15****.*

**36:00** I proved that the determinants are the same.

**39:30** I proved that the eigenvalues are the same.

**44:15** Did another 3x3 example. It’s not diagonalizable. But it’s nearly diagonalizable - this is the Jordan Canonical Form theorem. It’s not in the course but it’s super-important and you’ll likely use it before you graduate.

## Lecture 38: Best Approximate Solution of a Linear System

Nicholson Section 5.6

Watch the video on [YouTube](https://www.youtube.com/watch?v=5n6u-hsH73Y) or [MyMedia](https://play.library.utoronto.ca/2ef9d9c4153b007aad93b1cb6023e0a2)

**Video Duration:** 47:49

### Video Description:

Started by introducing the Gobstoppers and Stopgobbers problem; this is a smaller version of the backpacker’s problem. Demonstrated that there’s no exact solution.

**5:40** Introduced the concept of best approximate solution. Found the best approximate solution via projection onto the GS,SG subspace and then solving a related linear algebra problem. This is the “slow way” of finding the best approximate solution. *Note that in this lecture I produced the orthogonal basis for the subspace via “magic”; this was because the example had been given before the Gram-Schmidt process had been discussed. You’ve already learnt about Gram-Schmidt and so there’s nothing magical here; you know how to find that pair of vectors.*

**15:45** *(Note the change in clothing - it’s a different lecture!)* Reviewed the diet problem in which there are more equations than there are unknowns. There’s no solution. But is this the best we can do? 18:20 - Reviewed the concept of “best approximate solution of A$\rightharpoonaccent{x}$=$\rightharpoonaccent{b}$”. Reviewed the “slow way” of finding the approximate solution.

**21:50** What happens if you don’t first try to find the solution before proceeding to find the best approximate solution? What happens if there actually is a solution? Good news! The best approximate solution will turn out to be a solution in this case. So you don’t need to first try and find a solution of A$\rightharpoonaccent{x}$=$\rightharpoonaccent{b}$ first. *I guess the way to view this is that sometimes the best approximate solution involves no approximation at all; it’s actually the solution.*

**25:10** Introduced a faster and easier way to find the best approximate solution. Derived the linear system AT A $\rightharpoonaccent{x}$ = AT $\rightharpoonaccent{b}$.

**37:00** Applied this approach to the diet problem.

**43:35** Will AT A always be invertible?

**44:15** Applied best approximate solution to a data fitting example using data from the [Spurious Correlations webpage](http://tylervigen.com/spurious-correlations). Trying to find best linear fit to the data.

## Lecture 39: Introduction to Orthogonal Subspaces

Nicholson Section 8.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=ZdzH795ObWs) or [MyMedia](https://play.library.utoronto.ca/3c195470c74a130eee21a9cf4ff8adad)

**Video Duration:** 50:36

### Video Description:

Started with a review of what it means for a vector to be orthogonal to a subspace. Defined S-perp; the subspace of all vectors in Rn that are orthogonal to S.

**2:15** Did an example where S is the span of two vectors in R3. Geometrically, we know what S is and what S-perp should be. But how, in general, can we tell if a specific vector is orthogonal to S? This would involve computing infinitely many dot products! Demonstrated how it’s sufficient to simply test the vector against the vectors in a spanning set for the subspace.

**10:40** If $\rightharpoonaccent{v}$ is orthogonal to S then so is t$\rightharpoonaccent{v}$ for every scalar.

**13:30** Stated the theorem that if S is the span of k vectors in Rn then $\rightharpoonaccent{v} $is orthogonal to S if and only if $\rightharpoonaccent{v}$ is orthogonal to each of the k vectors in the spanning set. Proved theorem.

**18:30** Started focusing on S-perp. Found S-perp for the previous example of S. We’ve already found a set of vectors in S-perp. To show that this is all of S-perp, we need to show that every vector in S-perp is in that set of vectors. Showed how to formulate “please find S-perp” as a problem of the form A$\rightharpoonaccent{x}$=$\rightharpoonaccent{0}$. Used this to find S-perp.

**27:55** Computed S-perp where S is the span of 4 vectors in R4. This is a case where I don’t have geometric intuition about S or S-perp. I have to solve the problem algebraically and see what I learn from the process. Reviewed how to formulate “please find S-perp” as a problem of the form A$\rightharpoonaccent{x}$=$\rightharpoonaccent{0}$ and found S-perp. Note that in the process of finding S-perp, we found a basis for S, we found the dimension of S, we found a basis for S-perp, and we found the dimension of S-perp and we found that dim(S)+dim(S-perp)=4. *I made a mistake at* ***35:00****!! To find a basis of S, we need a basis for Row(A) because we put the spanning set of S into the rows of A. I wrote down a basis for Col(A), not row (A). Doh!*

**36:35** Presented a general approach to finding S-perp. Explained why dim(S-perp) = n-rank(A).

**39:40** Did example where S is a line in R3. Geometrically, expect S-perp to be the plane through the origin orthogonal to the line. Algebraically found S-perp. *This is one of those examples that students hate because you end up looking at a matrix with only one row and asking questions about its rank and so forth. Make sure you’re comfortable with this! Students don’t like matrices with only one row.*

**43:30** How to project a vector onto a subspace? We know how to project onto a line, but… what if we have a vector $\rightharpoonaccent{x}$ and we’d like to write $\rightharpoonaccent{x}$ as the sum of two vectors, one in S and the other in S-perp? Explained why we’d like to do this and reminded students that we’ve already done this in R2 and in R3.

**46:15** If I have a basis for S, can I write that the projection of $\rightharpoonaccent{x}$ onto S is the sum of the projections of $\rightharpoonaccent{x}$ onto each basis vector? NO! It fails if the basis isn’t an orthogonal set of vectors! Now the challenge is: given a basis for S, is there some way to transform this into an orthogonal basis for S?

## Lecture 40: How to project onto a subspace

Nicholson section 8.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=yWbld0OsxLU) or [MyMedia](https://play.library.utoronto.ca/8ce931a6131c7b87b1bacb0b95d984f4)

**Video Duration:** 48:50

### Video Description:

Started by reminding students of something from the previous lecture: why finding S-perp is the same as solving a problem of the form A$\rightharpoonaccent{x}$=$\rightharpoonaccent{0}$. Also, what’s cool is that in the process of doing this you get a basis for S for free! Fixed a mistake I’d made in the previous lecture: one finds a basis for S by taking the row space of A, not the column space of A.

**8:50** Did another example, this time where S is given as the span of two vectors in R5. Found S-perp and also found a basis for S.

**16:40** Stated that if S is a subspace of Rn then dim(S)+dim(S-perp)=n.

**17:35** Started discussion of projecting a vector onto a subspace.

**18:45** Reviewed how it works in R2. Proj\_S($\rightharpoonaccent{x}$) is defined to be the vector $\rightharpoonaccent{s0}$ in S that is the closest to $\rightharpoonaccent{x}$. In general, how do we find this vector $\rightharpoonaccent{s0}$?

**23:10** Stated and proved the theorem “If $\rightharpoonaccent{s0}$ is in S and if $\rightharpoonaccent{x}$ -$\rightharpoonaccent{s0}$ is orthogonal to S then $\rightharpoonaccent{s0}$ is the vector in S that is the closest to $\rightharpoonaccent{x}$.” This theorem gives us a way to find Proj\_S($\rightharpoonaccent{x}$)!

**33:10** Did an example where S is the span of two vectors in R4. Projected a specific vector onto S. Presented two different ways to find this projection: a fast way and a slow way. *The slow way will turn out to be useful for something else, so don’t completely ignore it.* J The fast way involves finding an orthogonal basis for S, projecting onto each basis vector, and adding up the projections. Which is great, if you have a way of finding an orthogonal basis for S.

## Lecture 41: Introduction to the Gram-Schmidt process

Nicholson Section 8.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=poG6CO8O4jE) or [MyMedia](https://play.library.utoronto.ca/68d413b2865efffba5a1d19e269895db)

**Video Duration:** 47:53

### Video Description:

Started class with a “why do we care about projections?” word problem - how much powdered milk, KD, and Gatorade should you take backpacking so as to most-closely satisfy the daily required diet.

**11:30** Reviewed that there were two different ways of projecting a vector onto a subspace and it’s super-easy if you happen to have an orthogonal basis for the subspace.

**14:00** What if you have an orthonormal basis for the subspace? This makes the formula for the projection prettier but it makes the vectors in the orthonormal basis look uglier.

**19:45** Given a basis, how can we create an orthogonal basis from it? I started with a simple example of two vectors in R2.

**25:00** What would have happened if I’d done that example but handled the vectors in different order?

**29:00** Considered S the span of three vectors in R3. I’d like to find an orthogonal basis for S. But at this point I don’t even have a basis for S! The good news: applying the procedure to a spanning set for S will produce an orthogonal basis for S (and we can then determine the dimension of S). In this particular example, it turns out that the original spanning set was a basis for S.

**42:15** I know that S = span of the given vectors because this is how I was given S. I then did a bunch of stuff to the spanning set. How do I know that the final set of vectors spans S? How do I know that I didn’t mess anything up? *I raised this question but didn’t answer it.*

**43:15** Did another example where S is the span of three vectors and I sought an orthogonal set of vectors that spans S.

## Lecture 42: Finishing up the Gram-Schmidt process

Nicholson Section 8.1

Watch the video on [YouTube](https://www.youtube.com/watch?v=SM-eGBzDs00) or [MyMedia](https://play.library.utoronto.ca/6c8080dd6078529a3b7fac9137172cbe)

**Video Duration:** 35:36

### Video Description:

Reviewed the Gram-Schmidt process. This time I wrote the process in terms of projecting onto subspaces and subtracting off those projections.

**10:20** Did Gram-Schmidt to a set of six vectors in R5, they span a subspace S. I’d like a basis for S and the dimension of S. What does it mean if you get zero vectors while applying the Gram-Schmidt process?

***20:00*** *I made a mistake here. I should have asked students to accept that Span{v1,v2,v3,v4,v5,v6} = Span{w1,w2,w3,w4,w5,w6}. The v6 is missing on the blackboard. Doh! It’s corrected by a student eventually but still…*

**22:50** Gave a warning that Gram-Schmidt is super on paper but if you actually implement it on a computer you’ll find that it’s numerically unstable and roundoff error messes stuff up. If you want to do it on a computer you should use the Modified Gram-Schmidt method or something even more sophisticated.

**24:50** I stated a theorem which will allow us to be confident about Gram-Schmidt not messing up the span at any step. Demonstrated how to use the theorem.

**31:10** Proved theorem for the special case of three vectors.

*If you’re curious about the modified Gram-Schmidt method and how it compares to the vanilla Gram-Schmidt method have a look at* [*Solving the Normal Equations by QR and Gram-Schmitd*](https://ocw.mit.edu/courses/mathematics/18-335j-introduction-to-numerical-methods-spring-2019/week-4/MIT18_335JS19_lec9_reading.pdf) *or* [*The modified Gram-Schmidt procedure*](https://www.math.uci.edu/~ttrogdon/105A/html/Lecture23.html)

*Here's a nice document (*[*Gram-Schmidt Orthogonalization: 100 Years and More*](http://www.cis.upenn.edu/~cis610/Gram-Schmidt-Bjorck.pdf)*) that includes some of this history behind Gram-Schmidt, modified Gram-Schmidt, Least-squares approximation (another way of describing our best approximate solutions) and an interesting sci-fi connection. The notes are from an advanced course so don’t expect to understand all 50 slides.*

## Lecture 43: Non-diagonalizable matrices examples; all symmetric matrices are diagonalizable

Set-up for Nicholson Section 8.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=Wp3O5sEhBE4) or [MyMedia](https://play.library.utoronto.ca/992360ab73ec83d341dda17bbe7392e6)

**Video Duration:** 37:20

**Video Description:**

Started with a T/F question “Every (square) matrix can be diagonalized”. *A better counter-example would have been the matrix [2 1;0 2] so that you can see the eigenvalues 2,2 as separate from the off-diagonal 1. It’s the off-diagonal 1 that stops the matrix from being diagonalizable.* What went wrong? If there’s even one eigenvalue for which the algebraic multiplicity is larger than the geometric multiplicity, the matrix will not be diagonalizable.

**10:50** None of the examples of non-diagonalizable matrices are symmetric. Is this a failure of imagination? Are there symmetric matrices that are non-diagonalizable? No. It’s a theorem - all symmetric matrices are diagonalizable.

**12:00** If A is upper triangular, lower triangular, or diagonal is it true that the eigenvalues are precisely on the diagonal? Yes. Did a 4x4 example to show why this is true - you should verify that it’s true in general.

**15:45** Diagonalized a symmetric 3x3 matrix.

**24:45** After all those computations, nothing had to do with A’s being symmetric. Here’s something interesting: the eigenvectors are mutually orthogonal! This is because A is symmetric. Demonstrated that PT P is a diagonal matrix. Modified the eigenvectors (made them unit vectors) and this made PT equal to P-1. (We diagonalized the matrix without having to compute P-1 using the inverse matrix algorithm.)

**34:30** Defined what it means for a square matrix to be orthogonally diagonalizable.

## Lecture 44: How to orthogonally diagonalize a symmetric matrix

Nicholson Section 8.2

Watch the video on [YouTube](https://www.youtube.com/watch?v=FK5ZuTh97jw) or [MyMedia](https://play.library.utoronto.ca/7d933cc8bbcb4ed751907d56b5bddb0b)

**Video Duration:** 51:37

### Video Description:

Started with an overview for the process of finding out if a matrix is diagonalizable.

**4:40** Turned the discussion to symmetric matrices.

**5:15** Quick review of the 3x3 symmetric example from previous lecture.

**8:50** Review of orthogonal diagonalization for this example.

**10:20** Why we care about orthogonal diagonalization at a physical level.

**12:35** Diagonalized another symmetric 3x3 matrix. Found the eigenvalues and the eigenvectors that one gets via the usual process.

**16:00**  Wrote down the 3 eigenvectors. Do they form an orthogonal set? No! That said, eigenvectors that have different eigenvalues are mutually orthogonal.

**17:15** We have a basis for the eigenspace; can we transform it into an orthogonal basis? Time for Gram-Schmidt! Using this, I find a different pair of eigenvectors from the original pair of eigenvectors. *Important thing for you to check: if I have vectors* $\rightharpoonaccent{x}$ *and* $\rightharpoonaccent{y}$ *and they’re both eigenvectors with eigenvalue l then any linear combination of* $\rightharpoonaccent{x}$ *and* $\rightharpoonaccent{y}$ *will also be an eigenvector with eigenvalue l. But if* $\rightharpoonaccent{x}$ *is an eigenvector with eigenvalue l and* $\rightharpoonaccent{y}$ *is an eigenvector with eigenvalue m where l¹m then linear combinations of* $\rightharpoonaccent{x}$ *and* $\rightharpoonaccent{y}$ *will not be eigenvectors unless they’re linear combinations where one (but not both) of the coefficients equals zero.*

**23:45** How the process presented at the beginning of class is modified for orthogonal diagonalization.

**25:20** Gave students the challenge problem of finding a symmetric matrix that has specific eigenspaces. *This is how the authors of textbooks generate the matrices in the exercises; how they have such nice eigenvalues and eigenvectors.*

**34:00** What happens to the eigenvalues of a matrix when you multiply the matrix by a scalar? The eigenvalues are multiplied by the same scalar. *This is important to remember - it’s a classic thing students get wrong.*

**37:00** Stated and proved the theorem that if A is symmetric and $\rightharpoonaccent{x}$ is an eigenvector with eigenvalue l and $\rightharpoonaccent{y}$ is an eigenvector with eigenvalue m where l¹m then $\rightharpoonaccent{x} $and $\rightharpoonaccent{y}$ are orthogonal.

**49:50** Finished class with three key T/F questions.