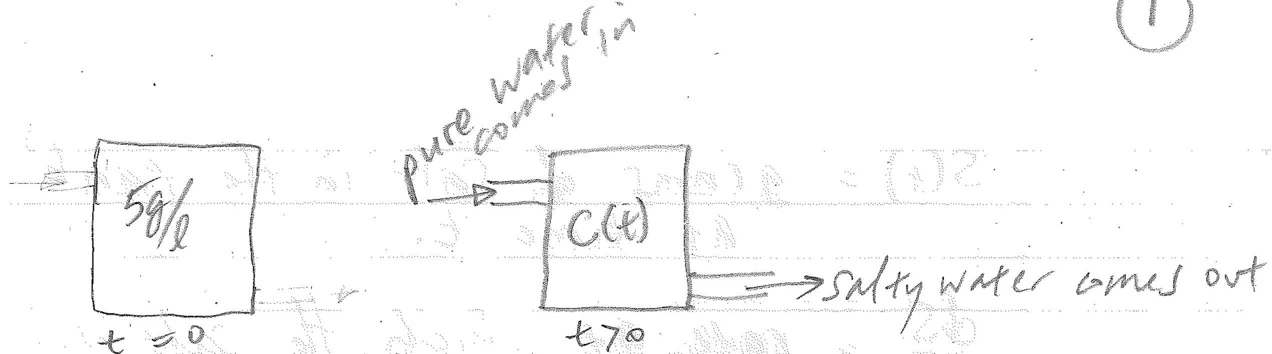


We have a 100 liter tank of salt water. The initial concentration is 5g/liter. At $t=0$, a tap is opened at the top and pure water comes in at 1 l/minute. At the same time, a tap is opened at the bottom & drains at 1 l/minute.

Q: What is the long time limit of the salt concentration in the tank?

Q: What is the salt concentration at time t?

Q: How would you choose the flow rate so that 100 minutes after the flows are turned on, the salt concentration is 2g/l?

We will assume that the fluids mix instantly and completely. This is an approximation: in reality the fluid near the top will be less salty than the fluid near the bottom.

Step 1: We can answer the first question without mathematics. Pure water is coming in so if we wait long enough the water in the tank will be less and less salty: $\lim_{t\to\infty} C(t) = 0$.

Step 2: We need to write down an equation for how the concentration changes in time.

$S(t) =$ grams of salt in the tank at time t.

$\frac{dS}{dt} =$ rate at which the salt is changing

$$\frac{dS}{dt} = \left(\begin{array}{c}\text{increase due to}\\ \text{salt coming in}\end{array}\right) - \left(\begin{array}{c}\text{decrease due}\\ \text{to salt leaving}\end{array}\right)$$

---

increase due to salt coming in $= 0$ because pure water is coming in

---

decrease due to salt leaving $= \dfrac{\text{current amount of salt}}{\text{current vol. of liquid}} \cdot$ drainage rate

$$= \frac{S(t)\text{ grams}}{100\text{ liter}} \times \frac{1\text{ liter}}{\text{minute}}$$

$$= \frac{S(t)}{100}\ \frac{\text{grams}}{\text{minute}}$$

(note: if the flow rate in didn't equal the flow rate out, the volume would depend on time and the denominator would be nonconstant in time.)

Combining,

$$\begin{cases} \dfrac{dS}{dt} = -\dfrac{1}{100} S(t) \\ S(0) = 5\dfrac{\text{grams}}{\text{liter}} \times 100\text{ liters} \\ \quad = 500\text{ grams} \end{cases}$$

Step 3: We need to solve the equation

And so we see that we seek a function $S(t)$ that satisfies

$$\begin{cases} \frac{dS}{dt}(t) = -\frac{1}{100} S(t) \text{ for } t > 0. \\ S(0) = 500 \text{ grams} \end{cases}$$

The only function that works is
$S(t) = 500\, e^{-t/100}$ grams

Check it out!

① Does it satisfy the initial data?

$S(0) = 500\, e^{-0/100} = 500$ grams

② Does it satisfy the differential equation?

$$\underbrace{\frac{dS}{dt}(t)}_{\text{left-hand side of eqn}} = \underbrace{-\frac{1}{100} S(t)}_{\text{right-hand side of eqn}}$$

left-hand side of eqn $= \frac{dS}{dt} = -\frac{1}{100}\left(500\, e^{-t/100}\right)$

(I just differentiated $S(t)$).

right hand side of eqn $= -\frac{1}{100}\left(500\, e^{-t/100}\right)$

(I plugged $S(t)$ into the RHS.)

does the left-hand side of the equation equal the right hand side for every t? Yes!

This shows that

$$S(t) = 500\, e^{-t/100}\ g \text{ solves}$$

the problem.

Step 4: Answer the question "What is the salt concentration at time t?

Answer: $C(t) = \frac{S(t) \text{ grams}}{100 \text{ liters}}$

$\lim_{t \to \infty} C(t) = 0$ as expected!

$$\boxed{C(t) = 5\, e^{-t/100}\ \frac{\text{grams}}{\text{liter}}}$$

Note: The concentration 100 minutes after the flows were turned on is

$$C(100) = 5\, e^{-100/100}\ g/l$$

$$\cong 1.84\ g/\text{liter}$$

Step 5: Answer "How would you choose the flow rate so that $C(100) = 2\, g/l$?"

From step 4, we know that the flow rate should be <u>less</u> than 1 liter/minute because

1.84 g/liter < 2 g/liter.

Going back to the derivation, let $r \triangleq$ the flow rate, we'll need to choose r later

$$\frac{dS}{dt} = -\frac{S(t) \text{ grams}}{100 \text{ liters}} \times r \frac{\text{liters}}{\text{minute}}$$

$$= -\frac{r}{100} S(t) \frac{\text{grams}}{\text{minute}}.$$

So we need to solve

$$\begin{cases} \frac{dS}{dt}(t) = -\frac{r}{100} S(t) & t > 0 \\ S(0) = 500 \end{cases}$$

This has solution

$$S(t) = 500\, e^{-\frac{r}{100}t} \text{ grams}$$

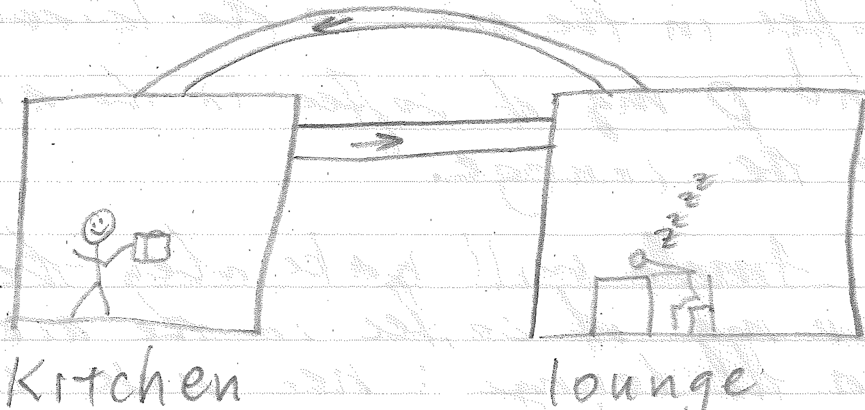
$$C(t) = 5\, e^{-\frac{r}{100}t} \text{ grams/liter}.$$

Choose r so that

$$C(100) = 5\, e^{-r} \text{ g/l} \quad \text{(plugged in } t=100\text{)}$$
$$= 2 \text{ g/l} \quad \text{(what I want)}$$

$$\Rightarrow e^{-r} = \frac{2}{5} \Rightarrow \boxed{r = \ln(5/2) \approx .92 \text{ l/minute}}$$

Smell the Coffee and Wake up!!



At time 0, Zsofia in the kitchen has made coffee. The air smells of coffee. Amala is asleep in the lounge. The kitchen fan sends air into the lounge at a rate of 20 l/minute. The lounge fan sends air into the kitchen at a rate of 20 l/minute.

The volume of the kitchen = the volume of the lounge = $100\ m^3$

1 liter $= .001\ m^3 \Rightarrow$ Kitchen volume

$$= 100\, m^3 \times \frac{1\ l}{.01\ m^3}$$

$$= 10{,}000\ l.$$

Assume air only circulates between the two rooms no other sources of air. (Poor Students!)

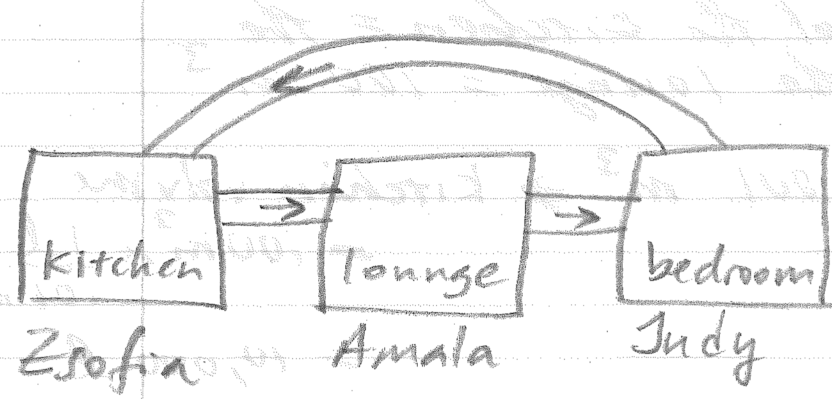
Assume the air mixes perfectly and instantly.

Q! At $t=0$, there are 100 mg of coffee in the air in the kitchen and 0 mg of coffee in the air in the lounge.

Amala will wake when there's 20 mg of coffee in the air in the lounge: her nose is set to alert at $\frac{20}{10{,}000}$ mg/l.

Q: When will Amala wake up?

Q: As $t \to \infty$ How much coffee will be in the air in each room?

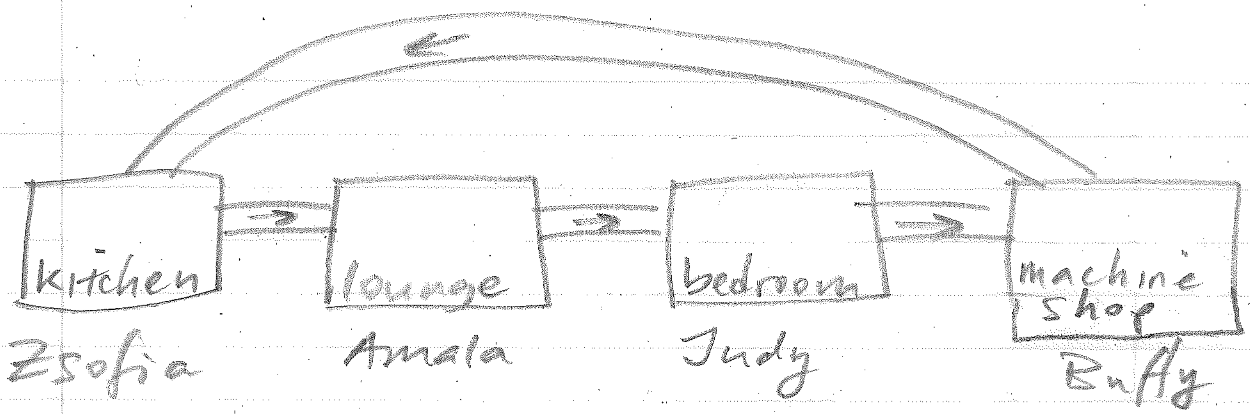


Same initial amount of coffee, same flow rates. Same room volumes

Q: When will Amala wake up?

Q: When will Judy wake up?

Q: As $t \to \infty$ how much coffee in each room?



Q: When will Amala wake up?

Q: When will Judy wake up?

Q: When will Buffy wake up?

Q: As $t \to \infty$, how much coffee in each room?

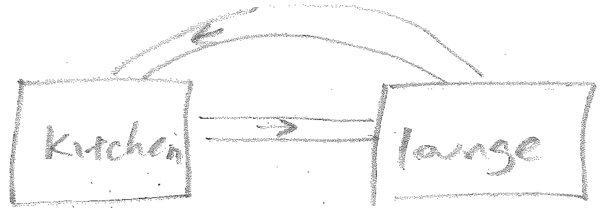
Bigger questions

How do your answers depend on the size of the room?

How does your answer depend on the flow rate?

What if it took 30 mg/l to wake Buffy up in the machine shop? Would she wake up?

Did Buffy turn the drill press off before falling asleep?



$x_1(t)$ = mg of coffee in kitchen air at time t

$x_2(t)$ = mg of coffee in air in loung at time t

$$\frac{dx_1}{dt} = \begin{pmatrix}\text{increase due}\\ \text{to coffee}\\ \text{coming from}\\ \text{lounge}\end{pmatrix} - \begin{pmatrix}\text{decrease due}\\ \text{to coffee}\\ \text{leaving kitchen}\\ \text{to lounge}\end{pmatrix}$$

$$= \frac{x_2(t)\text{ mg}}{10{,}000\ \ell} \cdot \frac{20\ \ell}{\text{min}} - \frac{x_1(t)}{10{,}000\ \ell} \cdot \frac{20\ \ell}{\text{min}}$$

So $\boxed{\dfrac{dx_1}{dt} = \dfrac{1}{500} x_2(t) - \dfrac{1}{500} x_1(t)}$

Similarly,

$$\frac{dx_2}{dt} = \begin{pmatrix}\text{increase due}\\ \text{to coffee}\\ \text{coming from}\\ \text{kitchen}\end{pmatrix} - \begin{pmatrix}\text{decrease due}\\ \text{to coffee}\\ \text{leaving lounge}\\ \text{to kitchen}\end{pmatrix}$$

$$= \frac{x_1(t)\text{ mg}}{10{,}000\ \ell} \cdot \frac{20\ \ell}{\text{min}} - \frac{x_2(t)\text{ mg}}{10{,}000\ \ell} \cdot \frac{20\ \ell}{\text{min}}$$

$$\boxed{\frac{dx_2}{dt} = \frac{1}{500} x_1(t) - \frac{1}{500} x_2(t)}$$

So we need to solve

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{1}{500}x_1 + \dfrac{1}{500}x_2 \\[2ex] \dfrac{dx_2}{dt} = \dfrac{1}{500}x_1 - \dfrac{1}{500}x_2 \end{cases}$$

with initial condition

$$\begin{cases} x_1(0) = 100 \text{ mg} \\ x_2(0) = 0 \text{ mg} \end{cases}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -1/500 & \frac{1}{500} \\ 1/500 & -1/500 \end{pmatrix} \vec{x} \qquad \boxed{\frac{d\vec{x}}{dt} = A\vec{x}}$$

seek eigenvalues + eigenvectors of

$$\begin{pmatrix} -1/500 & 1/500 \\ 1/500 & -1/500 \end{pmatrix}$$

$$\det \begin{pmatrix} -1/500 - \lambda & 1/500 \\ 1/500 & -\frac{1}{500} - \lambda \end{pmatrix}$$

$$= \left(-\frac{1}{500} - \lambda\right)\left(-\frac{1}{500} - \lambda\right) - \frac{1}{(500)^2}$$

$$= \lambda\left(\lambda + \frac{2}{500}\right)$$

eigenvectors?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda_1 = 0$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \lambda = -\frac{2}{500} = -\frac{1}{250}$$

General solution

$$\vec{x}(t) = c_1 e^{0t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t/250}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

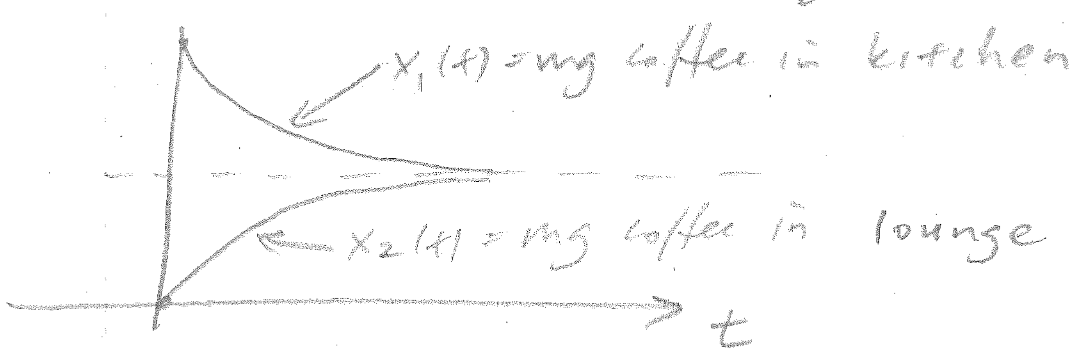
$$\Rightarrow c_1 = 50$$
$$c_2 = 50$$

$$\vec{x}(t) = 50\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 50\, e^{-t/250}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} x_1(t) = 50 + 50\, e^{-t/250} \\ x_2(t) = 50 - 50\, e^{-t/250} \end{cases}$$

$t \to \infty$ $\quad x_1(t) \to 50$ mg

$x_2(t) \to 50$ mg as expected



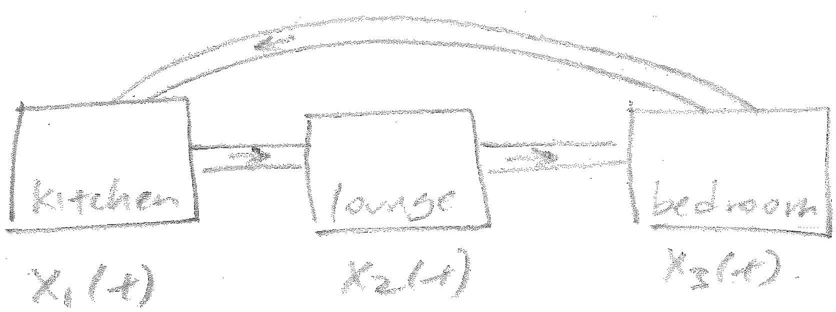
Amala wakes when $X_2(t) = 20$ mg

$$X(t) = 50 - 50e^{-t/250} = 20$$

we want

$$\frac{30}{50} = e^{-t/250}$$

$$t = 250 \ln(5/3) \approx 128 \text{ minutes.}$$



$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{1}{500} x_1(t) + \dfrac{1}{500} x_3(t) \\ \dfrac{dx_2}{dt} = -\dfrac{1}{500} x_2(t) + \dfrac{1}{500} x_1(t) \\ \dfrac{dx_3}{dt} = -\dfrac{1}{500} x_3(t) + \dfrac{1}{500} x_2(t) \end{cases}$$

with initial conditions

$x_1(0) = 100$
$x_2(0) = 0$
$x_3(0) = 0$

$$\vec{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{pmatrix} -\frac{1}{500} & 0 & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} & 0 \\ 0 & \frac{1}{500} & -\frac{1}{500} \end{pmatrix}$$

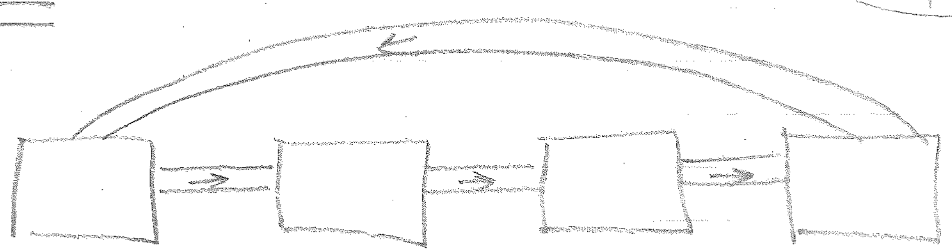
eigenvalues

$$\lambda_1 = 0 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1}{500}\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$\lambda_3 = \frac{1}{500}\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)$$

Complex eigenvalues! Not hard but will wait until 2nd year

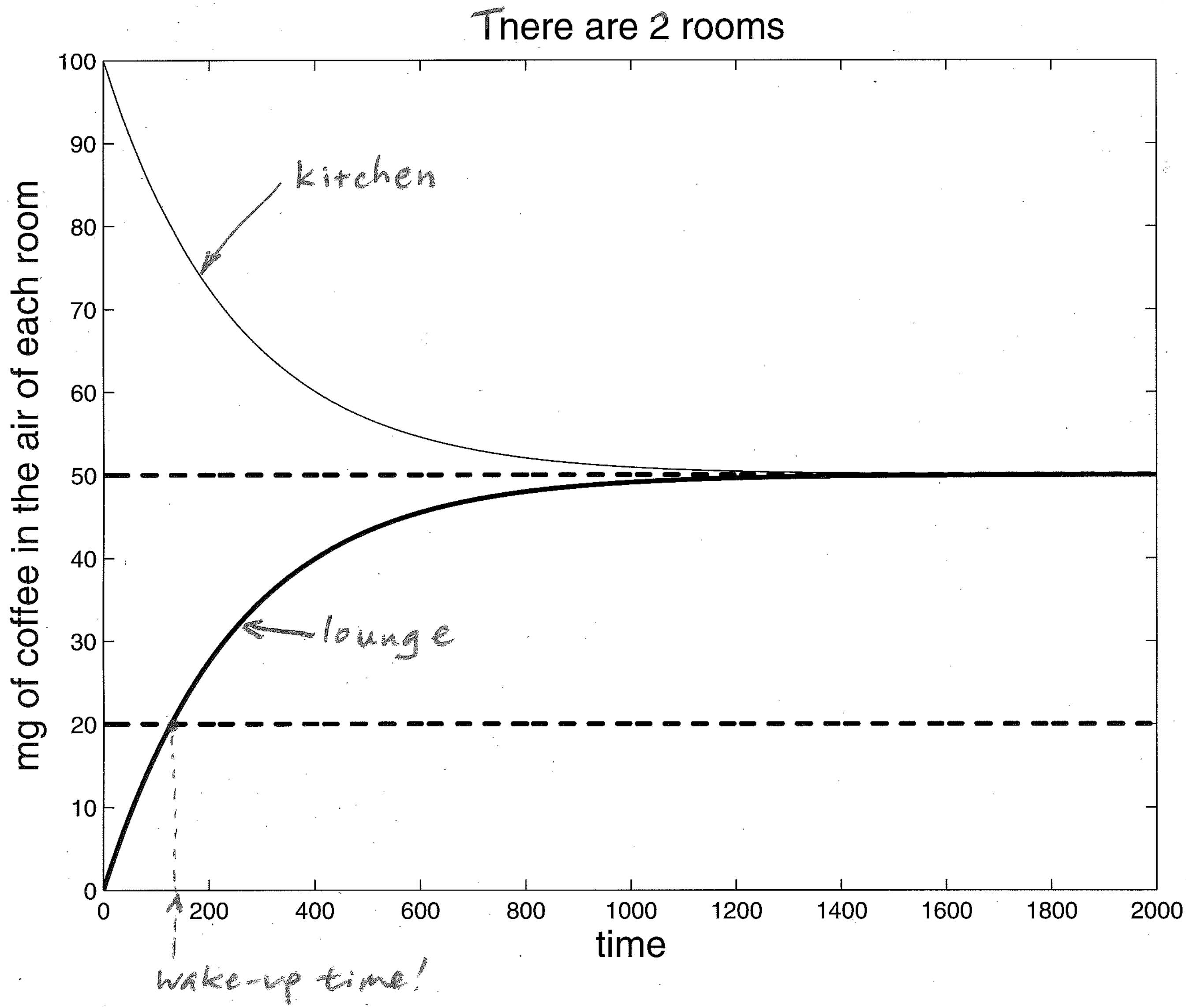


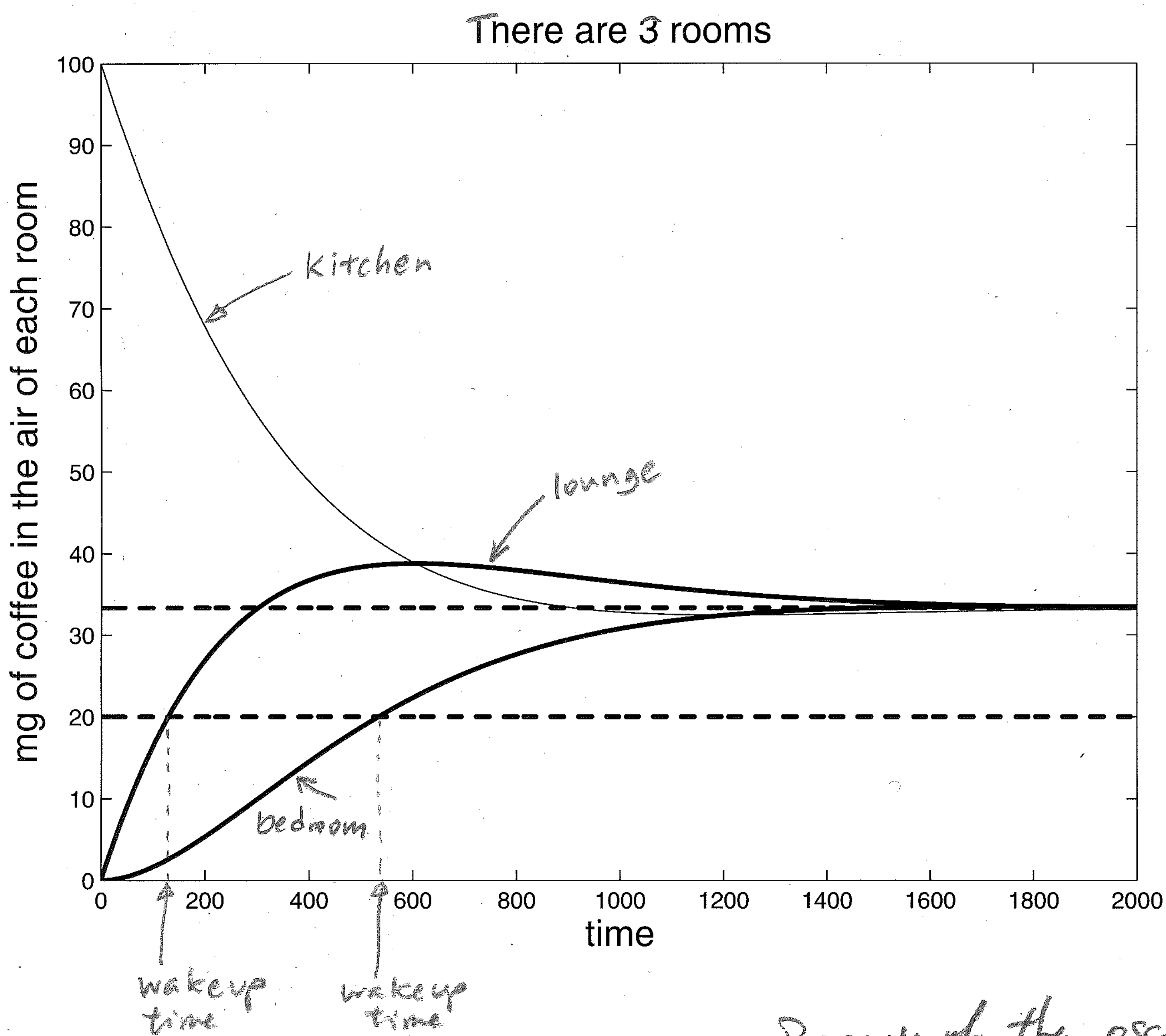
$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -\frac{1}{500} & 0 & 0 & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} & 0 & 0 \\ 0 & \frac{1}{500} & -\frac{1}{500} & 0 \\ 0 & 0 & \frac{1}{500} & -\frac{1}{500} \end{pmatrix} \vec{x}$$

$$\lambda_1 = 0 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

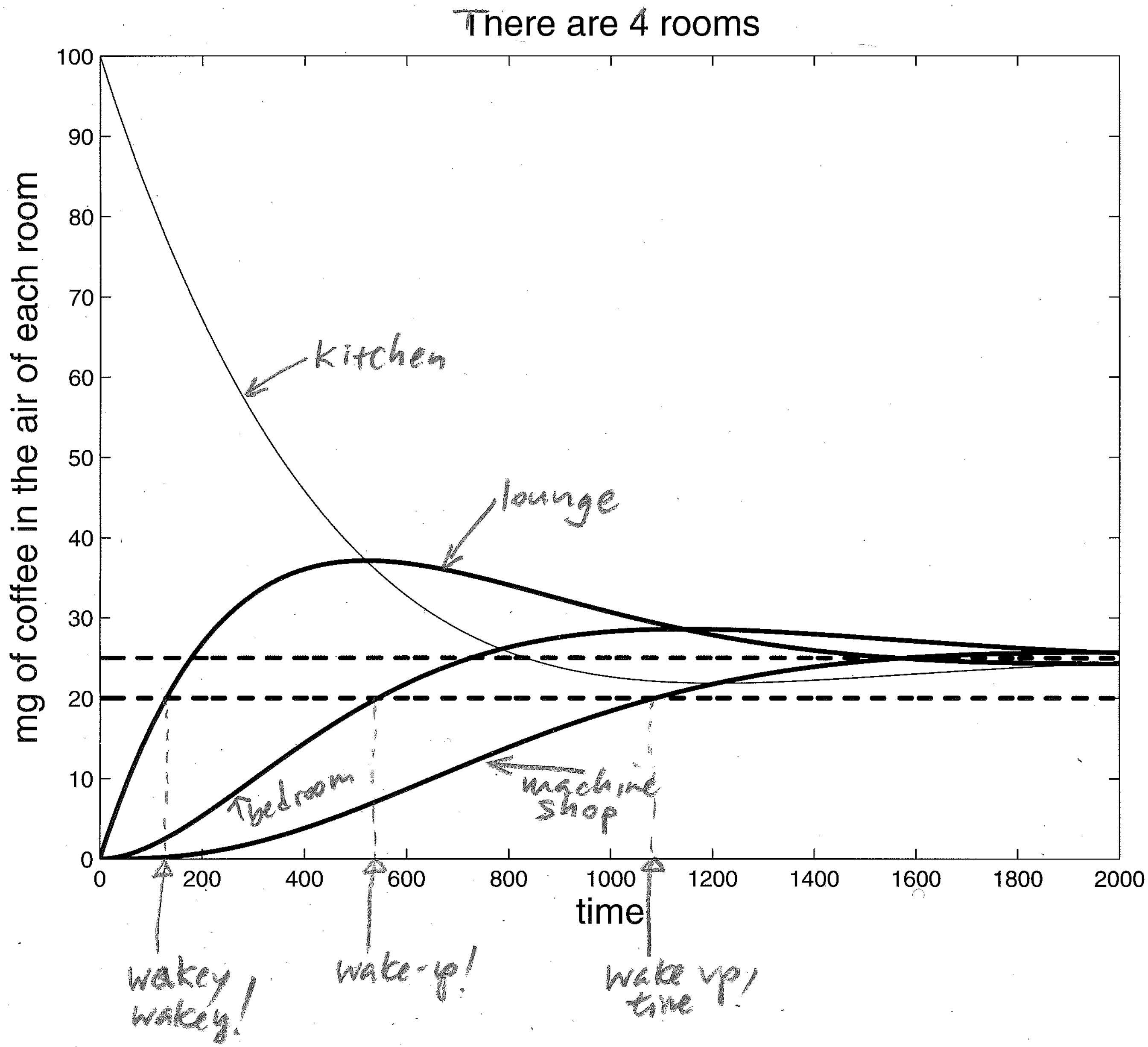
$$\lambda_2 = \frac{1}{500}(-1+i)$$

$$\lambda_3 = \frac{1}{500}(-1-i) \qquad \lambda_4 = -\frac{2}{500}$$


There are 2 rooms
mg of coffee in the air of each room
time
kitchen
lounge
wake-up time!



Because of the oscillatory parts of the solutions, there are times when there's more coffee in the lounge than in the kitchen. Also, there're times when there's more coffee in the bedroom than in the kitchen.



Some questions you can study...

1) The student in the lounge is the first to wake-up. How does her wake-up time depend on the number of rooms?

2) For $n > 5$, the wake-up amount (20 mg) is greater than the long-time limit amount ($100/n$). Will all students wake due to smelling coffee?

<Warning! The next 2 pages involve complex numbers! Skim, as needed!>

To solve the 3-room problem, we use the 3 eigenvalues and 3 eigenvectors

$$\lambda_1 = 0 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1}{500}\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right) \qquad \vec{v}_2 = \begin{pmatrix} -1/2 - i\sqrt{3}/2 \\ -1/2 + i\sqrt{3}/2 \\ 1 \end{pmatrix}$$ complex-valued eigenvector!

$$\lambda_3 = \frac{1}{500}\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) \qquad \vec{v}_3 = \begin{pmatrix} -\frac{1}{2} + i\sqrt{3}/2 \\ -\frac{1}{2} - i\sqrt{3}/2 \\ 1 \end{pmatrix}$$

The general solution is

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + c_3 e^{\lambda_3 t}\vec{v}_3$$

We choose c_1, c_2, c_3 by requiring the initial data be satisfied:

$$\begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

$$\Rightarrow \begin{cases} c_1 = \dfrac{100}{3} \\[2ex] c_2 = -\dfrac{100}{6} + \dfrac{50}{3}\sqrt{3}\,i \\[2ex] c_3 = -\dfrac{100}{6} - \dfrac{50}{3}\sqrt{3}\,i \end{cases}$$

Plugging these values in, and using that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\vec{x}(t) = \frac{100}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{100}{3} e^{-\frac{3}{1000}t}\cos\left(\frac{\sqrt{3}}{1000}t\right)\begin{pmatrix}2\\-1\\-1\end{pmatrix}$$
$$+\frac{100\sqrt{3}}{3} e^{-\frac{3}{1000}t}\sin\left(\frac{\sqrt{3}}{1000}t\right)\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$x_1(t) = \frac{100}{3} + \frac{200}{3} e^{-\frac{3}{1000}t}\cos\left(\frac{\sqrt{3}}{1000}t\right)$$

$$x_2(t) = \frac{100}{3} - \frac{100}{3} e^{-\frac{3}{1000}t}\cos\left(\frac{\sqrt{3}}{1000}t\right) + \frac{100\sqrt{3}}{3} e^{-\frac{3}{1000}t}\sin\left(\frac{\sqrt{3}}{1000}t\right)$$

$$x_3(t) = \frac{100}{3} - \frac{100}{3} e^{-\frac{3}{1000}t}\cos\left(\frac{\sqrt{3}}{1000}t\right) - \frac{100\sqrt{3}}{3} e^{-\frac{3}{1000}t}\sin\left(\frac{\sqrt{3}}{1000}t\right)$$

Note: as $t \to \infty$ $x_1(t) \to 100/3$
$x_2(t) \to 100/3$ $x_3(t) \to 100/3$
as expected

Note: To find when the first student will wake, we need to solve

$$x_2(t_{wake}) = 20 = \frac{100}{3} - \frac{100}{3} e^{-\frac{3}{1000}t_{wake}}\left[\cos\left(\frac{\sqrt{3}}{1000}t_{wake}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{1000}t_{wake}\right)\right]$$

To solve this, we need to use a computer.

<u>Note</u>: No matter how many rooms, if they're connected in a cycle, there will always be $\lambda_1 = 0$ and $\vec{v}_1 = [1;1;1;\ldots 1]$

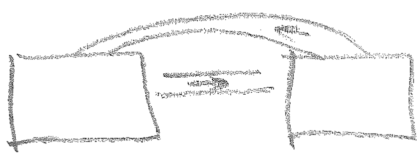
The other eigenvalues will have negative real parts and the solution with initial data $\vec{x}(0) = 100\,\vec{e}_1$ will have

$$\vec{x}(t) \to \frac{100}{n}[1;1;1;\ldots 1]$$

Where n = number of rooms. <u>Translation</u>

as time passes the coffee is more and more equally distributed in the rooms. As $t \to \infty$ $\vec{x}(t) \to$ the distribution of $\frac{100}{n}$ mg per room

<u>Note</u>: If the flow rate were r and the room volume were V then



would yield $\dfrac{d\vec{x}}{dt} = \begin{pmatrix} -\frac{r}{V} & \frac{r}{V} \\ \frac{r}{V} & -\frac{r}{V} \end{pmatrix} \vec{x}$

That is $A = \frac{r}{V}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ eigenvalues are...

$$\det\left(\frac{r}{V}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} - \lambda I\right) = \det\left(\frac{r}{V}\left[\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} - \frac{V}{r}\lambda I\right]\right)$$

$$= \left(\frac{r}{V}\right)^2 \det\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} - \frac{V}{r}\lambda I\right) = 0$$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ has eigenvalues $-2, 0 \Rightarrow A$ has eigenvalues $-\frac{2r}{V}, 0$

So the general solution of

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{with } A = \begin{pmatrix} -r/V & r/V \\ r/V & -r/V \end{pmatrix}$$

will be

$$\vec{x}(t) = 50e^{0t}\begin{pmatrix}1\\1\end{pmatrix} + 50e^{-\frac{2r}{V}t}\begin{pmatrix}1\\-1\end{pmatrix}$$

<u>trend</u>: the larger $\frac{2r}{V}$ is the faster $x_1(t)$ decreases to 50 mg and the faster $x_2(t)$ increases to 50 mg.

<u>trend</u>: the larger the flow rate (r) or the smaller the room size (V) the sooner Amala will wake up.

In general, however many rooms there are, if they're connected in a cycle, have the same room volume (V) and have the same flow rate between rooms (r) then the eigenvalues will be a multiple of r/V and we'll have the same effect on the rate of decay.

Theorem: if A is $n \times n$ and has eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$ and $B = tA$ then the eigenvalues of B are $t\lambda_1, t\lambda_2, \dots t\lambda_n$

Proof:

$C_A(\lambda) = \det(A - \lambda I) \qquad C_B(\mu) = \det(B - \mu I)$

$$
\begin{aligned}
t^n C_A(\lambda) &= t^n \det(A - \lambda I) \\
&= \det\big(t(A - \lambda I)\big) \\
&= \det(tA - t\lambda I) \\
&= \det(B - t\lambda I) && \text{because } B = tA \\
&= C_B(t\lambda) && \text{took } \mu = \lambda t \text{ in } C_B(\mu) = \det(B - \mu I)
\end{aligned}
$$

So $C_B(t\lambda) = t^n C_A(\lambda)$.

If λ_i is an eigenvalue of A then $C_A(\lambda_i) = 0 \Rightarrow t^n C_A(\lambda_i) = 0 \Rightarrow C_B(t\lambda_i) = 0$
$\Rightarrow t\lambda_i$ is an eigenvalue of B. This finishes the proof. //